Published by Institute of Physics Publishing for SISSA
Received: *December 22, 2006*
Accepted: *January 5, 2007*
Published: *January 22, 2007*

Permutation orientifolds of Gepner models

Kazuo Hosomichi

Service de Physique Théorique, CEA Saclay,
F-91191 Gif sur Yvette, France
E-mail: Kazuo.Hosomichi@cea.fr

Abstract: In tensor products of a left-right symmetric CFT, one can define permutation orientifolds by combining orientation reversal with involutive permutation symmetries. We construct the corresponding crosscap states in general rational CFTs and their orbifolds, and study in detail those in products of affine $\mathrm{U}(1)_2$ models or $N = 2$ minimal models. The results are used to construct permutation orientifolds of Gepner models. We list the permutation orientifolds in a few simple Gepner models, and study some of their physical properties — supersymmetry, tension and RR charges. We also study the action of corresponding parity on D-branes, and determine the gauge group on a stack of parity-invariant D-branes. Tadpole cancellation condition and some of its solutions are also presented.

Keywords: D-branes, Conformal Field Models in String Theory.

Contents

1. Introduction

In the construction of type II string vacua with $\mathcal{N} = 1$ supersymmetry in four dimensions, orientifolds play an important role along with branes and fluxes. While we wish to obtain a global picture for the whole variety of such vacua, it would be desirable to understand better each of the ingredients at different vacua. At one regime of vacua where the compactification manifold has large volume, the supergravity and Dirac Born-Infeld theory will give a reliable geometric description of the system. On the other hand, in a different regime where the size of the compactification manifold is very small, there are vacua admitting an exactly solvable worldsheet description. The worldsheet theory describing such vacua was found by Gepner [1] and involves an orbifold of products of $N = 2$ minimal models, which are very well-understood rational CFTs.

D-branes and orientifolds in Gepner models were studied in many papers. A class of D-branes were first constructed in [2] using Cardy's boundary states [3] in $N = 2$ minimal models. Since then, different aspects of them were studied including how they continue in moduli space to the large volume [4]. Similar analyses for orientifolds were first made in [5, 6] and then in [7]–[14] using the standard crosscap states in $N = 2$ minimal models, and provided us with a large number of tadpole-free backgrounds where the particle spectra are explicitly computable [15].

The D-branes and orientifolds studied in those works are mostly made from products of boundary or crosscap states in minimal models. On the other hand, in Gepner models containing products of minimal models of the same level, there are also D-branes and orientifolds corresponding to boundary conditions on fields twisted by permutation symmetries. Permutation branes in general CFTs were first constructed by Recknagel [16] by generalizing Cardy's standard prescription [3] (see also [17]). Some generalizations of it have been discussed in [18 − 20]. There have also been many work on permutation branes in Gepner models [21]–[25], some of which employ the description in terms of *matrix factorization* of Landau-Ginzburg superpotential [26, 27]. A natural extension of these developments will be to construct permutation orientifolds in a similar manner.

One of our goal in this paper is to give a general prescription to construct permutation orientifolds in tensor product CFTs as well as their orbifolds, generalizing the standard construction of crosscap states in RCFT given by [28] and developed further by [29]–[36].

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The other goal is to apply it to Gepner models and study type II string vacua made of permutation orientifolds. Accordingly, the paper is organized into two parts.

In section 2 we present our general construction of permutation orientifolds in RCFTs and orbifolds thereof. In section 3 we apply our prescription to the theory of n Dirac fermions, using the fact that the theory is related to the affine $\mathrm{U}(1)_2^{\otimes n}$ model by orbifolding. We pay particular attention to assigning Grassmann parity to states and operators so that the acnticommutativity of fermions is correctly reproduced. In a similar manner, we construct in section 4 the boundary and crosscap states in $N = 2$ minimal models preserving an $N = 2$ superconformal symmetry.

In section 5 we classify permutation orientifolds in Gepner models and write down their explicit form. The construction of permutation D-branes will also be given here although there have been a lot of works on it; in particular we discuss in full detail the properties of short orbit branes, i.e. branes in orbifolds which are not simply the sum over orbifold images. In section 6 we analyze further some physical properties of permutation orientifolds in Gepner models. We will find out how various orientifolds act on D-branes, and determine the gauge group on a stack of parity-invariant D-branes. We also analyze the condition of tadpole cancellation and some of its solutions. We conclude in section 7 with some brief remarks.

Note added. A part of the results presented in this paper was obtained independently by Brunner and Mitev [37]. We were informed of their work in progress at an early stage of our work.

Rudiments of one-loop amplitudes

Here we collect our convention for various one-loop amplitudes in string theory.

Cylinder. The one-loop of open string stretching between two D-branes is a cylinder. We parametrize the worldsheet by (σ, t) with $0 \le \sigma \le \pi,\ t \sim t + 2\pi l$ or a complex coordinate $z = \sigma + it$. The endpoints $\sigma = 0$ and π are on D-branes $\langle \mathcal{B}_0|$ and $|\mathcal{B}_\pi\rangle$ respectively. The D-branes are characterized by different boundary conditions on fields. We assume the worldsheet conformal field theory to have a symmetry generated by holomorphic currents $W(z), \widetilde{W}(\bar{z})$ with spin $S_W \in \frac{1}{2}\mathbb{Z}$, and assume that the currents with integer (half-odd-integer) spins are bosonic (resp. fermionic). We restrict our interest to the boundary states satisfying

$$\langle \mathcal{B}_0| \left(\widetilde{W}(\bar{z}) - e^{-i\pi S_W} W(z)\right)_{\sigma=0} \;=\; 0 \;=\; \left(\widetilde{W}(\bar{z}) - e^{i\pi S_W} W(z)\right)_{\sigma=\pi} |\mathcal{B}_\pi\rangle. \tag{1.1}$$

Let X be a symmetry of the theory. The open closed duality relates the overlap of boundary states in X-twisted sector and the trace over open string Hilbert space with weight X,

$${}^X\langle \mathcal{B}_0| e^{-\pi H_c/l} |\mathcal{B}_\pi\rangle^X \;=\; \mathrm{Tr}_{\mathcal{B}_0,\mathcal{B}_\pi}\left[(-)^F e^{-2\pi H_o l} X\right]. \tag{1.2}$$

The right hand side is formally calculated as the path integral on the cylinder with the fields $\phi(\sigma, t)$ obeying boundary conditions specified by D-branes and the periodicity along time,

$$\phi(\sigma, t) \;=\; X^{-1}\phi(\sigma, t + 2\pi l)X. \tag{1.3}$$

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If one is interested in summing over spin structures, it is convenient to introduce the indices NSNS$\pm$, RR$\pm$ to label different boundary conditions for fermionic currents W and $\widetilde{W}$,

$$ {}_{\mathrm{Y}\pm}\langle \mathcal{B}_0| \left(\widetilde{W}(\bar{z}) \mp e^{-i\pi S_W} W(z)\right) \;=\; 0 \;=\; \left(\widetilde{W}(\bar{z}) \mp e^{i\pi S_W} W(z)\right) |\mathcal{B}_\pi\rangle_{\mathrm{Y}\pm}. \qquad (1.4) $$

$\mathrm{Y} = \mathrm{NSNS}$ (RR) indicates that the fermionic fields are anti-periodic (periodic) along time t.

Möbius strip. If the theory on a strip has a parity symmetry exchanging fields at σ and $\pi - \sigma$, the one-loop of open string of width π and the periodicity along time ($t \sim t + 2\pi l$) twisted by the parity is a Möbius strip. The boundary states $\langle \mathcal{B}_0|$ and $|\mathcal{B}_\pi\rangle$ then have to be parity images of each other. We assume there is a "basic" involutive parity P acting on the currents as

$$ PW(\sigma,t)P \;=\; e^{-i\pi S_W}\widetilde{W}(\pi-\sigma,t), \qquad P\widetilde{W}(\sigma,t)P \;=\; e^{i\pi S_W}W(\pi-\sigma,t), \qquad (1.5) $$

and consider parities of the form gP, defined by combining P with various symmetries g acting locally on fields and symmetry currents. The Möbius strip amplitude associated to the parity gP is a trace over open string Hilbert space (1.2) with $X = gP$. Alternatively, it is given by a path integral on a strip of width $\pi/2$ and period $4\pi l$ bounded by a boundary and a crosscap states. The fields satisfy twisted periodicity along time,

$$ \phi(\sigma,t) \;=\; X^{-1}\phi(\sigma,t+4\pi l)X, \qquad X \equiv (gP)^2. $$

The fields obey the boundary condition specified by $\langle \mathcal{B}_0|$ at $\sigma = 0$, and the crosscap condition at $\sigma = \pi/2$,

$$ \phi(\tfrac{\pi}{2},t) \;=\; gP\phi(\tfrac{\pi}{2},t-2\pi l)Pg^{-1}, \qquad (1.6) $$

The corresponding crosscap state is denoted by $|gP\rangle$. The open-closed duality then tells

$$ \mathrm{Tr}_{\mathcal{B}_0,\mathcal{B}_\pi}[(-)^F e^{-2\pi H_o l} gP] \;=\; {}^X\langle \mathcal{B}_0|e^{-\pi H_c/4l}|gP\rangle^X \;=\; {}^X\langle(-)^F gP|e^{-\pi H_c/4l}|\mathcal{B}_\pi\rangle^X. \qquad (1.7) $$

The second equality tells how the boundary states are transformed under the parity. The additional factor $(-)^F$ in the definition of crosscap bra-state is because we define the bra and ket states to satisfy the crosscap conditions

$$ \begin{aligned} 0 &= \langle gP| \left(\widetilde{W}(t) - e^{-i\pi S_W} gW(t-2\pi l)g^{-1}\right)_{\sigma=\frac{\pi}{2}} \\ &= \left(\widetilde{W}(t) - e^{i\pi S_W} gW(t-2\pi l)g^{-1}\right)_{\sigma=\frac{\pi}{2}} |gP\rangle, \end{aligned} \qquad (1.8) $$

so that (i) the conditions on bra and ket states are related by rotation by 180 degrees, and (ii) the bra and ket states are related by the dagger operation.

Different spin structures give a pair of NSNS crosscaps $|(-)^{F_L}P\rangle, |(-)^{F_R}P\rangle$ and a pair of RR crosscaps $|(\pm)^F P\rangle$ for each involutive parity symmetry P. In general the NSNS parity maps a boundary state ${}_{\mathrm{NSNS}\pm}\langle\mathcal{B}|$ to $|\mathcal{B}'\rangle_{\mathrm{NSNS}\pm}$ by (1.7), while the RR parity maps ${}_{\mathrm{RR}\pm}\langle\mathcal{B}|$ to $|\mathcal{B}'\rangle_{\mathrm{RR}\mp}$.

If $|P\rangle$ is the crosscap state corresponding to the parity P of (1.5), then $g|P\rangle$ satisfies

$$ \left(g\widetilde{W}(t)g^{-1} - e^{i\pi S_W} gW(t-2\pi l)g^{-1}\right)_{\sigma=\frac{\pi}{2}} g|P\rangle \;=\; 0. $$

We can therefore put

$$ g|P\rangle \;=\; |gPg^{-1}\rangle. \qquad (1.9) $$

Klein bottle. Let us next consider a closed string with spatial coordinate $\sigma \sim \sigma + 2\pi$. The one-loop of closed string with the periodicity along time ($t \sim t+2\pi l$) twisted by parity is a Klein bottle. If the parity maps σ to $-\sigma$ modulo 2π, then the Klein bottle is equivalent to a periodic strip of width π, period $t \sim t+4\pi l$ bounded by two crosscap states at $\sigma = 0$ and π. If the two crosscaps correspond to different parities $g_0 P$ and $g_\pi P$, then the fields obey

$$\phi(\sigma, t) \;=\; g_0 P_{(0)} \phi(\sigma, t-2\pi l) P_{(0)} g_0^{-1} \;=\; g_\pi P_{(\pi)} \phi(\sigma, t-2\pi l) P_{(\pi)} g_\pi^{-1}, \tag{1.10}$$

where the suffix (0) or (π) indicates the fixed point of the parity. Therefore the closed string is in the sector twisted by $g \equiv (g_0 g_\pi^{-1})$. The open-closed duality then tells that

$$\mathrm{Tr}_g[(-)^F e^{-2\pi H_{cl} l} g_0 P_{(0)}] \;=\; \mathrm{Tr}_g[(-)^F e^{-2\pi H_{cl} l} g_\pi P_{(\pi)}] \;=\; \langle (-)^F g_0 P | e^{-\frac{\pi H_c}{2l}} | g_\pi P \rangle. \tag{1.11}$$

The closed string states form a representation of the symmetry algebra of the currents $W(z)$ and $\widetilde{W}(\bar{z})$. The action of parity $P_{(0)}, P_{(\pi)}$ on the currents are given by (1.5) with modified fixed points. Introducing the coordinate $\zeta \equiv \pm e^{-iz}$ and expanding the currents in standard power series, one finds these parities act on the modes as $W_n \leftrightarrow \widetilde{W}_n$, as expected.

2. Permutation branes and crosscaps in RCFT

In this section we present the construction of permutation branes and orientifolds in tensor products of general rational CFTs, and then extend it to their simple current orbifolds. The argument follows that of [16].

Let $\mathcal{X}$ be a general left-right symmetric RCFT with chiral symmetry algebra $\mathcal{A} \otimes \mathcal{A}$, and denote the tensor product of N copies of it by $\mathcal{X}^N$. The D-branes or orientifolds in $\mathcal{X}$ are described by the states $|\mathcal{B}\rangle$, $|\mathcal{C}\rangle$ satisfying the boundary or crosscap conditions on currents generating two copies of $\mathcal{A}$:

$$\begin{aligned} (\widetilde{W}_n - e^{-i\pi S_W} W_{-n}) |\mathcal{B}\rangle^{\mathcal{X}} &= 0, \\ (\widetilde{W}_n - e^{-i\pi (S_W - n)} W_{-n}) |\mathcal{C}\rangle^{\mathcal{X}} &= 0. \end{aligned} \tag{2.1}$$

Here S_W is the spin of the current W. Any product of states $|\mathcal{B}\rangle$ or $|\mathcal{C}\rangle$ of $\mathcal{X}$ gives a state of $\mathcal{X}^N$ satisfying the boundary or crosscap conditions

$$\begin{aligned} (\widetilde{W}^a_n - e^{-i\pi S_W} W^a_{-n}) |\mathcal{B}\rangle^{\mathcal{X}^N} &= 0, \\ (\widetilde{W}^a_n - e^{-i\pi (S_W - n)} W^a_{-n}) |\mathcal{C}\rangle^{\mathcal{X}^N} &= 0. \end{aligned} \tag{2.2}$$

Here the suffix a is for operators in the a-th copy of $\mathcal{X}$. Permutation branes and permutation orientifolds in $\mathcal{X}^N$ are characterized by the conditions on currents twisted by permutations $\pi \in S_N$:

$$\begin{aligned} (\widetilde{W}^{\pi(a)}_n - e^{-i\pi S_W} W^a_{-n}) |\mathcal{B}^\pi\rangle^{\mathcal{X}^N} &= 0, \\ (\widetilde{W}^{\pi(a)}_n - e^{-i\pi (S_W - n)} W^a_{-n}) |\mathcal{C}^\pi\rangle^{\mathcal{X}^N} &= 0. \end{aligned} \tag{2.3}$$

We call these conditions as "π-permuted".

2.1 Cardy and Pradisi-Sagnotti-Stanev's constructions

In the standard Cardy and Pradisi-Sagnotti-Stanev(PSS) constructions, D-branes and orientifolds in general RCFT $\mathcal{X}$ are expressed as suitable linear sums of Ishibashi states which form the basis of solutions to (2.1). Here we extend this prescription to construct permutation branes and orientifolds in $\mathcal{X}^N$, following the argument of [16]. Our construction of permutation orientifolds agrees with that of [37].

General Ishibashi states $|\mathcal{B};i\rangle\rangle$ and $|\mathcal{C};i\rangle\rangle$ in $\mathcal{X}$ are constructed as

$$|\mathcal{B};i\rangle\rangle \;:=\; \sum_{M\in\mathcal{V}_i} |M\rangle\otimes\Phi|M\rangle, \qquad |\mathcal{C};i\rangle\rangle \;:=\; e^{\pi i(L_0-h_i)}\,|\mathcal{B};i\rangle\rangle. \tag{2.4}$$

Here $\mathcal{V}_i$ is the i-th highest weight representation of $\mathcal{A}$ spanned by an orthonormal basis $\{|M\rangle\}$, and h_i is its conformal weight. Φ is the anti-unitary operator satisfying $W_n\Phi = e^{-i\pi S_W}\Phi W^\dagger_{-n}$. The simple products of Ishibashi states $|\mathcal{B};i_1\cdots i_N\rangle\rangle$, $|\mathcal{C};i_1\cdots i_N\rangle\rangle$ satisfy the boundary or crosscap conditions (2.3) in $\mathcal{X}^N$ with $\pi = \mathrm{id}$. Define an operator R^π acting only on the left-moving (= antiholomorphic) operators and primary states as permutations

$$\begin{aligned} R^\pi \widetilde{W}^a_n R^{\pi^{-1}} &= \widetilde{W}^{\pi(a)}_n, \\ R^\pi\cdot|i_1\otimes\tilde{\imath}_1\rangle_1\cdots|i_N\otimes\tilde{\imath}_N\rangle_N &= (\pm)|i_1\otimes\tilde{\imath}_{\pi^{-1}(1)}\rangle_1\cdots|i_N\otimes\tilde{\imath}_{\pi^{-1}(N)}\rangle_N. \end{aligned} \tag{2.5}$$

Note that, in the second equation, R^π should be understood to annihilate the state unless the state $|i_a\otimes\tilde{\imath}_{\pi^{-1}(a)}\rangle$ is contained in the Hilbert space of $\mathcal{X}$ for all a. The $\pm$ sign in the right hand side of the second equation arises if the theory $\mathcal{X}$ contains fermionic states and currents. The π-permuted Ishibashi states are then simply given by

$$\begin{aligned} |\mathcal{B}^\pi;i_1\cdots i_N\rangle\rangle &= R^\pi\,|\mathcal{B};i_1\cdots i_N\rangle\rangle, \\ |\mathcal{C}^\pi;i_1\cdots i_N\rangle\rangle &= R^\pi\,|\mathcal{C};i_1\cdots i_N\rangle\rangle. \end{aligned} \tag{2.6}$$

In the rest of this subsection we assume $\mathcal{X}$ to be an RCFT with charge conjugation modular invariant, so that R^π annihilates the primary state $|i_1\otimes\bar{\imath}_1\rangle_1\cdots|i_N\otimes\bar{\imath}_N\rangle_N$ unless $i_{\pi^{-1}(a)} = i_a$ for all a. We also assume, for simplicity, that all the states and currents in $\mathcal{X}$ are bosonic. We denote the number of cycles in a given permutation π by $[\pi]$, the c-th cycle of π as π_c and its length by $\|\pi_c\|$. The π-permuted Ishibashi states can then be labelled by j_c $(c = 1,\cdots,[\pi])$ such that

$$i_a = j_c \;\text{ if } a\in\pi_c. \tag{2.7}$$

So we introduce another expression for Ishibashi states:

$$\begin{aligned} |\mathcal{B}^\pi;j_1\cdots j_{[\pi]}\rangle\rangle &= \otimes^{[\pi]}_{c=1}|\mathcal{B}^{\pi_c};j_c\rangle\rangle = \sum_{i_a}\delta^{(\pi)}_{i,j}\,R^\pi\,|\mathcal{B};i_1\cdots i_N\rangle\rangle, \\ |\mathcal{C}^\pi;j_1\cdots j_{[\pi]}\rangle\rangle &= \otimes^{[\pi]}_{c=1}|\mathcal{C}^{\pi_c};j_c\rangle\rangle = \sum_{i_a}\delta^{(\pi)}_{i,j}\,R^\pi\,|\mathcal{C};i_1\cdots i_N\rangle\rangle. \end{aligned} \tag{2.8}$$

The delta symbol $\delta^{(\pi)}_{i,j}$ enforces the condition (2.7). The inner products of these Ishibashi states read

$$
\begin{aligned}
\langle\!\langle \mathcal{B}^{\tilde\pi}, \tilde{j}_1\cdots\tilde{j}_{[\pi]}|\, e^{\pi i\tau H}\, |\mathcal{B}^{\pi}; j_1\cdots j_{[\pi]}\rangle\!\rangle &= \sum_{i,j'} \delta^{(\tilde\pi)}_{i,\tilde j}\delta^{(\pi)}_{i,j}\delta^{(\sigma)}_{i,j'}\prod_{c=1}^{[\sigma]}\chi_{j'_c}(\|\sigma_c\|\tau),\\
\langle\!\langle \mathcal{C}^{\tilde\pi}, \tilde{j}_1\cdots\tilde{j}_{[\pi]}|\, e^{\pi i\tau H}\, |\mathcal{C}^{\pi}; j_1\cdots j_{[\pi]}\rangle\!\rangle &= \sum_{i,j'} \delta^{(\tilde\pi)}_{i,\tilde j}\delta^{(\pi)}_{i,j}\delta^{(\sigma)}_{i,j'}\prod_{c=1}^{[\sigma]}\chi_{j'_c}(\|\sigma_c\|\tau),\\
\langle\!\langle \mathcal{B}^{\tilde\pi}, \tilde{j}_1\cdots\tilde{j}_{[\pi]}|\, e^{\pi i\tau H}\, |\mathcal{C}^{\pi}; j_1\cdots j_{[\pi]}\rangle\!\rangle &= \sum_{i,j'} \delta^{(\tilde\pi)}_{i,\tilde j}\delta^{(\pi)}_{i,j}\delta^{(\sigma)}_{i,j'}\prod_{c=1}^{[\sigma]}\mathcal{I}^{\|\sigma_c\|}\chi_{j'_A}(\|\sigma_c\|\tau),\\
&\qquad (\sigma\equiv\pi^{-1}\circ\tilde\pi)
\end{aligned}
\tag{2.9}
$$

where $\mathcal{I}$ is an involutive operator defined to act on characters as

$$
\mathcal{I}\chi_i(\tau) = \widehat{\chi}_i(\tau) \equiv e^{-\pi i(h_i-\frac{c}{24})}\chi_i(\tau+1/2). \tag{2.10}
$$

D-branes and orientifolds are linear combinations of Ishibashi states satisfying certain consistency conditions. Recknagel [16] constructed the permutation branes as follows:

$$
|\mathcal{B}^{\pi}_{\mathbf{J}}\rangle \;=\; |\mathcal{B}^{\pi}_{J_1\cdots J_{[\pi]}}\rangle \;=\; \bigotimes_{c=1}^{[\pi]}|\mathcal{B}^{\pi_c}_{J_c}\rangle \;=\; \bigotimes_{c=1}^{[\pi]}\sum_{j_c}\frac{S_{J_cj_c}}{(S_{0j_c})^{\|\pi_c\|/2}}|\mathcal{B}^{\pi_c}; j_c\rangle\!\rangle. \tag{2.11}
$$

In [16] it was also shown that the open string spectrum between any two such D-branes satisfies integrality. To see this, let us consider the finest possible decomposition of the set of N letters, $\{1,\cdots,N\}=\bigcup_b \mathcal{S}_b$ such that any cycle of $\pi,\tilde\pi$ or $\sigma=\pi^{-1}\circ\tilde\pi$ is contained in one of $\mathcal{S}_b$. The annulus amplitude then becomes

$$
\begin{aligned}
\langle\mathcal{B}^{\tilde\pi}_{\tilde{\mathbf{J}}}|e^{-\pi H/l}|\mathcal{B}^{\pi}_{\mathbf{J}}\rangle &= \sum_{J'_1,\cdots,J'_{[\sigma]}}\prod_b \mathcal{N}_b(\tilde{\mathbf{J}},\mathbf{J},\mathbf{J}')\prod_{c=1}^{[\sigma]}\chi_{J'_c}(il/\|\sigma_c\|),\\
\mathcal{N}_b(\tilde{\mathbf{J}},\mathbf{J},\mathbf{J}') &= \sum_j\prod_{\tilde\pi_c\in\mathcal{S}_b}\frac{S^*_{\tilde J_cj}}{(S_{0j})^{\|\tilde\pi_c\|/2}}\prod_{\pi_c\in\mathcal{S}_b}\frac{S_{J_cj}}{(S_{0j})^{\|\pi_c\|/2}}\prod_{\sigma_c\in\mathcal{S}_b}S_{J'_cj}.
\end{aligned}
\tag{2.12}
$$

The coefficient $\mathcal{N}_b$ always takes the form

$$
\begin{aligned}
&\sum_j\frac{S_{J_1j}\cdots S_{J_{n+3}j}}{S_{0j}^{2g+n+1}}\quad (n\geq 0,\ g\geq 0)\\
&=\begin{cases}[N_{J_2}N_{J_3}\cdots N_{J_{n+2}}]_{J_1}^{\;\bar J_{n+3}} & (g=0)\\ \sum_{j_1,\cdots,j_g}\mathrm{Tr}[N_{J_1}N_{J_2}\cdots N_{J_{n+3}}\cdot N_{j_1}N_{\bar j_1}\cdots N_{j_{g-1}}N_{\bar j_{g-1}}] & (g>0)\end{cases}
\end{aligned}
\tag{2.13}
$$

where N_i is the fusion matrix whose elements are all non-negative integers,

$$
[N_j]_k^{\;l} \;=\; N_{jk}^l \;=\; \sum_i\frac{S_{ji}S_{ki}S^*_{li}}{S_{0i}}.
$$

Hence $\mathcal{N}_b$ is always a nonnegative integer. The right hand side of (2.13) has an interpretation as the number of $(n+3)$-point conformal blocks on genus-g Riemann surface.

The construction of [16] can be extended to crosscap states in a straightforward manner. General permutation orientifold of $\mathcal{X}^N$ is labelled by an involutive permutation π and a parity $P_I \equiv \otimes_{a=1}^N P_{I_a}$ satisfying $P_{I_{\pi(a)}} = P_{I_a}$. Then we propose the following crosscap states,

$$|\mathcal{C}^\pi_{\mathbf{I}}\rangle \;=\; |\mathcal{C}^\pi_{I_1\cdots I_{[\pi]}}\rangle \;=\; \bigotimes_{c=1}^{[\pi]} |\mathcal{C}^{\pi_c}_{I_c}\rangle = \bigotimes_{c=1}^{[\pi]} \sum_{j_c} \frac{X_{I_c j_c}}{(S_{0j_c})^{\|\pi_c\|/2}} |\mathcal{C}^{\pi_c}; j_c\rangle\rangle, \tag{2.14}$$

$$X_{I_c j_c} = \begin{cases} P_{I_c j_c} \;\; (\|\pi_c\| = 1) \\ S_{I_c j_c} \;\; (\|\pi_c\| = 2) \end{cases}$$

Note that the lengths of all the cycles of π have to be one or two for π to be involutive. The integrality of Klein bottle and Möbius strip amplitudes can be checked by a direct computation. One encounters factors of the form

$$\sum_j \frac{S_{J_1 j} \cdots S_{J_{m+1} j} P_{I_1 j} \cdots P_{I_{2n} j}}{S_{0j}^{2l+m+2n-1}} \qquad (m,n,l \geq 0,\; m+2n \geq 2), \tag{2.15}$$

which can be rewritten in a similar way as (2.13), using the N- and Y-matrices

$$[Y_j]_k{}^l \;=\; Y^l_{jk} \;=\; \sum_i \frac{S_{ji} P_{ki} P^*_{li}}{S_{0i}}, \tag{2.16}$$

whose elements are all known to be integers. For this rewriting to be possible, the number of P-matrices in (2.15) has to be always even; this is actually the case because we put $X_{I_c j_c} = P_{I_c j_c}$ or $S_{I_c j_c}$ depending on $\|\pi_c\| = 1$ or 2. To check this, let us consider the Klein bottle amplitudes between π- and $\tilde{\pi}$-permuted crosscap states. In order to write them down one needs the decomposition $\{1, \cdots, N\} = \bigcup_b \mathcal{S}_b$ in the same way as for the annulus amplitudes. The factors of the form (2.15) are associated to each of $\mathcal{S}_b$. One finds the number of P-matrices in (2.15) is the sum of the numbers of odd-length cycles of π and those of $\tilde{\pi}$ contained in $\mathcal{S}_b$, which is always even. The same argument applies to Möbius strip amplitudes.

In summary, for an RCFT $\mathcal{X}$ defined with charge conjugation modular invariant, the formulae (2.11) and (2.14) give general π-permuted boundary and crosscap states in $\mathcal{X}^N$.

2.2 Simple current orbifold

Here we briefly review some basic properties of simple current orbifolds $\mathcal{X}/G$ and the constructions of D-branes and orientifolds in such theories.

Suppose a CFT $\mathcal{X}$ has a group G of simple currents. A simple current $g \in G$ is by definition a representation of $\mathcal{A}$ which maps any representation into another unique representation under fusion:

$$g \times i \to gi.$$

It follows that g induces (infinitely many) invertible maps between two highest weight representations $\mathcal{V}_i, \mathcal{V}_{gi}$ of $\mathcal{A}$. For an RCFT $\mathcal{X}$ defined with charge conjugation modular invariant, its orbifold $\mathcal{X}/G$ is defined by the modular invariant

$$Z^{\mathcal{X}/G} = \frac{1}{|G|} \sum_i \sum_{g_1, g_2 \in G} e^{2\pi i (Q_{g_2}(i) - q(g_1, g_2))} \chi_i(\tau) \chi_{g_1 \bar{i}}(-\bar{\tau}). \tag{2.17}$$

Here $Q_g(i)$ is defined and characterized by

$$\begin{array}{ll} (1)\ Q_g(i) = h_i + h_g - h_{gi} & (\text{mod } \mathbb{Z}), \\ (2)\ Q_g(i) + Q_{g'}(i) = Q_{gg'}(i) & (\text{mod } \mathbb{Z}), \\ (3)\ Q_g(i) + Q_g(j) = Q_g(k) \ \text{ if } N_{ij}^k \neq 0 & (\text{mod } \mathbb{Z}), \end{array} \tag{2.18}$$

and $q(g_1, g_2)$ is a symmetric bilinear function of the elements of G satisfying

$$\begin{array}{ll} (4)\ Q_{g_1}(g_2) = 2q(g_1, g_2) & (\text{mod } \mathbb{Z}), \\ (5)\ q(g, g) = -h_g & (\text{mod } \mathbb{Z}). \end{array} \tag{2.19}$$

Modular invariance of $Z^{\mathcal{X}/G}$ follows from the above conditions together with an important formula [38, 39]:

$$S_{ij} e^{2\pi i Q_g(j)} = S_{gi,j}. \tag{2.20}$$

In the RCFT terms, the sector twisted by $g \in G$ of the orbifold theory $\mathcal{X}/G$ consists of the representation spaces $\mathcal{V}_i \otimes \mathcal{V}_{g\bar{i}}$ of $\mathcal{A} \otimes \mathcal{A}$. The ground state in this sector has the eigenvalue

$$g' \ = \ e^{2\pi i (Q_{g'}(i) - q(g, g'))}, \tag{2.21}$$

as can be read off from (2.17). In a formal field theory terms, each term in the torus partition function (2.17) of the orbifold theory $\mathcal{X}/G$ is given by the path integral of the fields $\phi(z)$ on a torus ($z \sim z + 2\pi \sim z + 2\pi\tau$) with the periodicity conditions

$$\phi(z) \ = \ g_1^{-1} \phi(z + 2\pi) g_1 \ = \ g_2^{-1} \phi(z + 2\pi\tau) g_2. \tag{2.22}$$

2.2.1 The issue of doubled periodicity

Although the function $Q_g(i)$ only needs to be defined modulo $\mathbb{Z}$ in constructing the modular invariant torus partition function, we wish to have it defined modulo $2\mathbb{Z}$ for constructing boundary or crosscap states in later sections. In what follows we assume that $Q_g(i)$ is defined modulo $2\mathbb{Z}$ so as to satisfy the equations (2,3,4) of (2.18)–(2.19) modulo $2\mathbb{Z}$, namely it is bilinear in g and i modulo $2\mathbb{Z}$. However, $Q_g(i)$ so defined will not always be single-valued (=periodic) modulo $2\mathbb{Z}$. For example, $\prod_a g_a = \text{id}$ does not necessarily lead to $\sum_a Q_{g_a}(i) = 0$ modulo $2\mathbb{Z}$, although the equality always holds modulo $\mathbb{Z}$. In later sections, this kind of subtlety will be called “doubled periodicity”.

In constructing crosscap states in orbifolds, we will also need to find an improvement of conformal weights

$$h_i \ \to \ h_i - \theta(i), \tag{2.23}$$

by an integer-valued function $\theta(i)$ so that the equations (1) and (5) hold modulo $2\mathbb{Z}$ as well. Again, the function $\theta(i)$ will not in general be single valued as a function of representation label i.

2.2.2 Branes

Boundary states in orbifolds $\mathcal{X}/G$ are constructed by summing over images and twists. Pick a boundary state $|\mathcal{B}_J\rangle$ in $\mathcal{X}$, and let $H \subset G$ be the stabilizer of J. Then there are boundary states in $\mathcal{X}/G$ in one to one correspondence with the characters ρ of its untwisted stabilizer $U \subset H$ [31, 32],

$$|\mathcal{B}_J^\rho\rangle^{\mathcal{X}/G} \;=\; \frac{\sqrt{|H|}}{\sqrt{|G||U|}} \sum_{g\in G/H,\; h\in U} g|\mathcal{B}_J\rangle^h\, \rho(h), \tag{2.24}$$

Here $|\mathcal{B}_J\rangle^h$ is the boundary state in the h-twisted sector and defined to satisfy

$$^h\langle\mathcal{B}_{J'}|e^{-\pi H_c/l} g|\mathcal{B}_J\rangle^h \;=\; \mathrm{Tr}_{J',gJ}[he^{-2\pi H_o l}], \tag{2.25}$$

i.e. their overlaps should be proportional to the traces over open string Hilbert space with additional weight h. It is important that the twist h does not run over all the elements in H. The definition of untwisted stabilizer group will be given in section 2.3. To construct the boundary states in orbifolds explicitly, one therefore needs the expression for the states $|\mathcal{B}_J\rangle^h$ in terms of Ishibashi states,

$$|\mathcal{B}_J\rangle^h \;=\; \sum_j \frac{S^{(h)}_{Jj}}{\sqrt{S_{0j}}} |\mathcal{B}; h(j)\rangle\!\rangle^h. \tag{2.26}$$

Here the matrix $S^{(h)}$ has indices J, j which run only over representations fixed by h, and the elements are supposed to satisfy

$$S^{(h)}_{g(J),j} \;=\; S^{(h)}_{J,j} \exp 2\pi i(Q_g(j) + q(g,h)). \tag{2.27}$$

2.2.3 Orientifolds

Crosscap states in $\mathcal{X}/G$ are constructed as sums of crosscaps in $\mathcal{X}$. Here we review the construction of [36].

Let P_I be an involutive parity symmetry of $\mathcal{X}$ and $|P_I\rangle$ the corresponding crosscap state. The parity P_I maps a state in $\mathcal{V}_j \otimes \mathcal{V}_{\bar{j}}$ to a state in $\mathcal{V}_{\bar{j}} \otimes \mathcal{V}_j$. For any $g \in G$, gP_I defines a parity whose action is that of P_I followed by the phase multiplication (2.21). gP_I is also involutive due to $gP_I = P_I g^{-1}$ which one can easily check. So there are crosscaps $|gP_I\rangle$ satisfying

$$\langle g\tilde{g}P_I|e^{-\pi H_c/l}|gP_I\rangle \;=\; \mathrm{Tr}_{\tilde{g}}[g\tilde{g}P_I^{(0)}e^{-\pi H_c l}] \;=\; \mathrm{Tr}_{\tilde{g}}[gP_I^{(\pi)}e^{-\pi H_c l}], \tag{2.28}$$

where the trace in the right hand side is over the $\tilde{g}$-twisted closed string states, and the superscripts $(0), (\pi)$ indicate the fixed points of the parity on the circle of circumference 2π. The crosscap $|P_I\rangle^{\mathcal{X}/G}$ in the orbifold is therefore described by a sum of crosscaps in $\mathcal{X}$,

$$|P_I\rangle^{\mathcal{X}/G} \;=\; \frac{1}{\sqrt{|G|}} \sum_{g\in G} |gP_I\rangle. \tag{2.29}$$

One can also consider the sum of crosscaps in $\mathcal{X}$ dressed by characters of G,

$$|P_I^\epsilon\rangle^{\mathcal{X}/G} \;=\; \frac{1}{\sqrt{|G|}}\sum_{g\in G}|gP_I\rangle\epsilon(g). \tag{2.30}$$

Note here that, since $g|P_I\rangle = |gP_Ig^{-1}\rangle = |g^2P_I\rangle$ from (1.9), the character ϵ in (2.30) have to be $\mathbb{Z}_2$-valued if the crosscap states in orbifold are made of G-invariant closed string states. Such a degree of freedom arises only when G contains an element of even order, i.e. if G/G^2 is non-trivial.

To extend the PSS construction to orbifolds, one needs to find the precise relation (including the normalization) between the crosscap state $|gP_I\rangle$ corresponding to the parity gP_I and the PSS state

$$|\mathcal{C}_{gI}\rangle = \sum_j \frac{P_{gI,j}}{\sqrt{S_{0j}}}\,|\mathcal{C};j\rangle\rangle.$$

From the formula for overlaps of two PSS states,

$$\begin{aligned}\langle\mathcal{C}_{g\tilde{g}I}|e^{-\pi H_c/l}|\mathcal{C}_{gI}\rangle &= \sum_j Y_{j,gI}^{\;g\tilde{g}I}\chi_j(il)\\ &= \sum_j Y_{j,I}^{\;\tilde{g}I}\chi_j(il)e^{\pi i\{h_{gI}+h_{\tilde{g}I}-h_{g\tilde{g}I}-h_I-2Q_g(j)\}},\end{aligned} \tag{2.31}$$

one finds that, for an arbitrary character $e^{i\pi\Delta(g)}$ of G, the following sum of PSS states

$$\frac{1}{\sqrt{|G|}}\sum_{g\in G}|\mathcal{C}_{gI}\rangle\exp\pi i\left\{h_I - q(g,g) - h_{gI} - \Delta(g)\right\}, \tag{2.32}$$

corresponds to a parity symmetry of the theory $\mathcal{X}/G$ which acts as

$$P_I\exp i\pi\left\{h_{gI}+q(g,g)-h_I+\Delta(g)\right\}$$

on g-twisted sector. The crosscap state (2.32) is G-invariant provided $\Delta(g^2) = 2Q_g(I)$ modulo $2\mathbb{Z}$, as follows from the identity [32, 33]

$$e^{2\pi iQ_g(j)}P_{i,j} \;=\; P_{g^2i,j}\exp i\pi(2h_g+2h_{gi}-h_i-h_{g^2i}). \tag{2.33}$$

We have thus found that, in order to define a parity P_I and the corresponding crosscap state in orbifold $\mathcal{X}/G$ from those in $\mathcal{X}$, we need to choose a character $e^{i\pi\Delta}$ of G satisfying $\Delta(g^2) = 2Q_g(I)$ mod $2\mathbb{Z}$. We find it most convenient to set $\Delta(g) = Q_g(I)$ mod $2\mathbb{Z}$, although this gives rise to some subtleties because $e^{i\pi\Delta}$ is actually not always a character of G.

We first notice that there exists an integer-valued function θ on the set of representations of $\mathcal{A}$ with the following property:[1]

$$h_I - q(g,g) - h_{gI} \;=\; Q_g(I)+\theta(I)-\theta(gI) \mod 2, \tag{2.34}$$

[1] G is assumed to act on I freely.

Putting $I :=$ id and setting $\theta(\text{id}) = 0$, one finds $\theta(g) = h_g + q(g, g)$. Inserting this back into (2.34) one finds that $\theta(I)$ can be thought of as a modification of h_I discussed at (2.23). Introducing $\sigma_I \equiv e^{i\pi\theta(I)}$, the requirement that (2.29) coincides with (2.32) up to an overall sign when $\Delta(g) = Q_g(I)$ just boils down to

$$|gP_I\rangle \;=\; |\mathcal{C}_{gI}\rangle\sigma_{gI}. \tag{2.35}$$

The general crosscap state in $\mathcal{X}/G$ is thus given by

$$|P_I^\epsilon\rangle^{\mathcal{X}/G} \;=\; \frac{1}{\sqrt{|G|}}\sum_{g\in G}|\mathcal{C}_{gI}\rangle\sigma_{gI}\cdot\epsilon(g). \tag{2.36}$$

The parity P_I^ϵ corresponding to this crosscap acts on g-twisted sector as $P_I\epsilon(g)\sigma_I\sigma_{gI}$.

The crosscaps $|gP_I\rangle$ defined by (2.35) satisfies the shift relation $g|P_I\rangle = |g^2P_I\rangle$, so the crosscap state (2.36) is a G-invariant closed string state. However, $|gP_I\rangle$ has in general doubled periodicity because of the doubled periodicity of σ_{gI}. Therefore, ϵ in (2.36) should be chosen in such a way that the summand in the right hand side is a single-valued function of $g \in G$.

2.3 Permutation branes in orbifolds

In this and the next subsections we consider the permutation branes and orientifolds in the orbifold $\mathcal{X}^N/\mathcal{G}$, where $\mathcal{G}$ is a subgroup of G^N. For simplicity, we assume $\mathcal{G}$ is invariant under S_N, namely,

$$g \equiv (g_1, \cdots, g_N) \in \mathcal{G} \implies g_\pi \equiv (g_{\pi(1)}, \cdots, g_{\pi(N)}) \in \mathcal{G}. \tag{2.37}$$

D-branes in $\mathcal{X}^N/\mathcal{G}$ are constructed as sums over images and twists. The simple current $g = (g_1, \cdots, g_N)$ acts on π-permuted boundary states $|\mathcal{B}^\pi_{\mathbf{J}}\rangle$ in $\mathcal{X}^N$ as

$$g|\mathcal{B}^\pi_{\mathbf{J}}\rangle \;=\; g\otimes_{c=1}^{[\pi]}|\mathcal{B}^{\pi_c}_{J_c}\rangle \;=\; \otimes_{c=1}^{[\pi]}|\mathcal{B}^{\pi_c}_{J'_c}\rangle, \qquad J'_c = (\textstyle\prod_{a\in\pi_c} g_a)J_c. \tag{2.38}$$

In particular, g fixes the brane $|\mathcal{B}^\pi_{\mathbf{J}}\rangle$ if

$$J_c = \textstyle\prod_{a\in\pi_c} g_a\cdot J_c \qquad c = 1, \cdots, [\pi].$$

As a simple example, all the π-permuted branes are fixed by g if $\prod_{a\in\pi_c} g_a = 1$ for all cycles π_c. Let us denote by $\mathcal{H} \subset \mathcal{G}$ the stabilizer of $|\mathcal{B}^\pi_{\mathbf{J}}\rangle$. Then the corresponding permutation brane in the orbifold takes the form [31, 32]

$$|\mathcal{B}^{\pi,\rho}_{\mathbf{J}}\rangle^{\mathcal{X}^N/\mathcal{G}} = \frac{|\mathcal{H}|}{\sqrt{|\mathcal{G}||\mathcal{U}|}}\sum_{h\in\mathcal{U}}\sum_{g\in\mathcal{G}/\mathcal{H}} g|\mathcal{B}^\pi_{\mathbf{J}}\rangle^h\,\rho(h), \tag{2.39}$$

where $|\mathcal{B}^\pi_{\mathbf{J}}\rangle^h$ denotes the boundary state in h-twisted sector. The twist h runs over the group $\mathcal{U} \subset \mathcal{H}$ called the untwisted stabilizer (see below for the definition) of the brane, and ρ is a character of $\mathcal{U}$.

The permutation boundary states in twisted sectors are constructed as follows. Since they factorize into pieces representing each cycle,

$$|\mathcal{B}^{\pi}_{\mathbf{J}}\rangle^{h} \;=\; \otimes_{c=1}^{[\pi]} |\mathcal{B}^{\pi_c}_{J_c}\rangle^{h}, \tag{2.40}$$

we focus on the cases where π itself is a cyclic permutation, $\pi = (1\,2\cdots N)$. For such π the boundary states in the sector twisted by $h = (h_1, \cdots, h_N)$ are defined by

$$|\mathcal{B}^{\pi}_{J}\rangle^{h} \;=\; \sum_{j} \frac{S^{(h_{\rm tot})}_{Jj}}{(S_{0j})^{N/2}} |\mathcal{B}^{\pi}; j\rangle\!\rangle^{h}, \tag{2.41}$$

where the matrix $S^{(h)}$ was introduced in (2.26), $h_{\rm tot} \equiv h_1 h_2 \cdots h_N$ and the Ishibashi states in h-twisted sector are defined by

$$\begin{aligned} |\mathcal{B}^{\pi}; j\rangle\!\rangle^{h} \;&\equiv\; R^{\pi} |\mathcal{B}; j_1 \cdots j_N\rangle\!\rangle, \\ j_k = h_k j_{k-1} \quad &(k = 1, \ldots, N;\; j_0 \equiv j). \end{aligned} \tag{2.42}$$

Note that the Ishibashi states defined in this way depend on the choice of the “first” entry in the cycle. For more general cyclic permutation $\pi = (a_1 a_2 \cdots a_N)$ we define $|\mathcal{B}^{\pi}; j\rangle\!\rangle^{h}$ so that $\bar{\jmath}$ appears in the a_1-th antiholomorphic sector and $h_{\rm tot} j = j$ appears in the a_N-th holomorphic sector.

In order for the sum over twisted sectors to make sense, we need to require that the J-label of $|\mathcal{B}^{\pi}_{J}\rangle^{h}$ is transformed in the same way as that of $|\mathcal{B}^{\pi}_{J}\rangle^{h={\rm id}}$ by simple currents:

$$g|\mathcal{B}^{\pi}_{J}\rangle^{h} \;=\; |\mathcal{B}^{\pi}_{g_{\rm tot}(J)}\rangle^{h} \omega_{\pi}(g,h). \quad (g_{\rm tot} \equiv g_1 g_2 \cdots g_N) \tag{2.43}$$

The factor $\omega_{\pi}(g,h)$, if nontrivial, means that $g \in \mathcal{G}$ not only acts on the J-label of the brane $|\mathcal{B}^{\pi,\rho}_{J}\rangle$ but also transforms $\rho(h)$ to $\rho(h)\omega_{\pi}(g,h)$. The simple current prescription gives

$$\begin{aligned} \omega_{\pi}(g,h) = \exp 2\pi i \,\{&-q(g_1 \cdots g_N, h_1 \cdots h_N) \\ &+ q(g_1, h_1) + q(g_2, h_1^2 h_2) + \cdots + q(g_N, h_1^2 h_2^2 \cdots h_{N-1}^2 h_N)\}\,. \end{aligned} \tag{2.44}$$

For a state $|\mathcal{B}^{\pi}_{J}\rangle^{h}$ in h-twisted sector to contribute to (2.39), $\mathcal{H}$ should be realized trivially on it; otherwise it would be projected out by the orbifolding procedure. The *untwisted stabilizer group* $\mathcal{U} \subset \mathcal{H}$ of a boundary state is formed by such h's. $\mathcal{U}$ is therefore formed by those $h \in \mathcal{H}$ satisfying $\omega_{\pi}(g,h) = 1$ for all $g \in \mathcal{H}$.

2.3.1 Diagonal branes

An interesting class of permutation D-branes are the “diagonal branes” in $\mathcal{X}^2$ or its orbifolds, which are regarded as wrapping the diagonal, $\mathcal{X} \subset \mathcal{X}^2$.

First, let us consider the following boundary state in the product theory $\mathcal{X}^2$,

$$|\mathcal{B}_{\rm diag}\rangle^{\mathcal{X}^2} \;\equiv\; |\mathcal{B}^{(12)}_{0}\rangle \;=\; \sum_{i} R^{(12)} |\mathcal{B}; i, i\rangle\!\rangle. \tag{2.45}$$

Note that the modular S-matrices in the enumerator and denominator of Recknagel's construction canceled out. It gives the annulus partition function,

$$ {}^{\mathcal{X}^2}\langle \mathcal{B}_{\rm diag}|e^{-\pi H_c/l}|\mathcal{B}_{\rm diag}\rangle^{\mathcal{X}^2} \;=\; \sum_i \chi_i(i/l)\chi_i(i/l) \;=\; \sum_i \chi_i(il)\chi_{\bar{\imath}}(il) \;=\; Z^{\mathcal{X}}_{T^2}(il). \quad (2.46) $$

Let us next consider an orbifold $\mathcal{X}^2/\mathcal{G}$. For simplicity, we take $\mathcal{G} = G\otimes G = \{(g_1, g_2)|g_1, g_2 \in G\}$ with G acting on all the representations in the theory $\mathcal{X}$ freely. The diagonal brane is invariant under the elements $h \otimes h^{-1} \in \mathcal{G}$, so we consider the sum over $h \otimes h^{-1}$-twisted sectors,

$$ \begin{aligned} |\mathcal{B}_{\rm diag}\rangle^{\rm orb} &= \frac{1}{\sqrt{|\mathcal{G}|}} \sum_{g,h\in G} (g\otimes 1)|\mathcal{B}_{\rm diag}\rangle^{h\otimes h^{-1}} \\ &= \frac{1}{\sqrt{|\mathcal{G}|}} \sum_{g,h\in G}\sum_i (g\otimes 1)R^{(12)}\,|\mathcal{B}; h(i), i\rangle\!\rangle. \end{aligned} \quad (2.47) $$

This diagonal brane gives the annulus partition function,

$$ \begin{aligned} {}^{\rm orb}\langle \mathcal{B}_{\rm diag}|e^{-\pi H_c/l}|\mathcal{B}_{\rm diag}\rangle^{\rm orb} &= \frac{1}{|G|}\sum_{g,h,i} e^{2\pi i Q_g(i)+2\pi i q(g,h)}\chi_{h(i)}(i/l)\chi_i(i/l) \\ &= \frac{1}{|G|}\sum_{g,h,j} e^{2\pi i Q_h(j)+2\pi i q(g,h)}\chi_j(il)\chi_{g^{-1}(\bar{\jmath})}(il) \;=\; Z^{\mathcal{X}/G}_{T^2}(il). \end{aligned} \quad (2.48) $$

Let us reconsider the properties of diagonal branes in more abstract terms. We first consider the product theory $\mathcal{X}^2$ defined on a strip of width π parametrized by $(\sigma \in [0,\pi], t \in \mathbb{R})$. We wish to consider what boundary condition on the fields $\phi_{1,2}$ corresponds to the diagonal brane. Suppose that the theory $\mathcal{X}$ on a circle ($\sigma \sim \sigma + 2\pi$) has an involutive parity symmetry P which acts linearly on fields ϕ as

$$ P \;:\; \phi(\sigma) \;\mapsto\; \mathcal{R}(P)\phi(-\sigma), \quad (2.49) $$

where $\mathcal{R}(P)$ is a matrix representation of P when ϕ is a vector describing the collection of fields. Then consider the theory $\mathcal{X}^2$ on a strip with the following boundary condition on fields at $\sigma = 0, \pi$:

$$ \begin{aligned} \phi_1(0) &= \mathcal{R}(P)\phi_2(0), & \phi_1(\pi) &= \mathcal{R}(P)\phi_2(\pi), \\ \phi_2(0) &= \mathcal{R}(P)\phi_1(0), & \phi_2(\pi) &= \mathcal{R}(P)\phi_1(\pi). \end{aligned} \quad (2.50) $$

One can then define a periodic field ϕ of the theory $\mathcal{X}$ on a circle of radius 2π by

$$ \begin{aligned} \phi(\sigma) &= \phi_1(\sigma) & &(\sigma \in [0,\pi]), \\ \phi(\sigma) &= \mathcal{R}(P)\phi_2(2\pi - \sigma) & &(\sigma \in [\pi, 2\pi]). \end{aligned} \quad (2.51) $$

The theory $\mathcal{X}^2$ on a strip with boundary condition (2.50) is thus equivalent to the theory $\mathcal{X}$ on a periodic cylinder. We therefore identify the fundamental diagonal branes $|\mathcal{B}_{\rm diag}\rangle$, $\langle\mathcal{B}_{\rm diag}|$ with the boundary conditions (2.50) on fields.

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Let us next consider the orbifold theory. We first wish to show that the overlap of $\langle\mathcal{B}_{\rm diag}|$ and $(g_1\otimes g_2)|\mathcal{B}_{\rm diag}\rangle$ gives a toroidal partition function of the theory $\mathcal{X}$ with periodicity along the σ direction twisted by $g_1^{-1}g_2^{-1}$. In field theoretic terms, the multiplication of $(g_1\otimes g_2)$ corresponds to the modification of the boundary condition on fields at $\sigma=\pi$,

$$\begin{aligned} g_1\phi_1 g_1^{-1} &= g_2(\mathcal{R}(P)\phi_2)g_2^{-1}, \\ g_2\phi_2 g_2^{-1} &= g_1(\mathcal{R}(P)\phi_1)g_1^{-1}. \end{aligned} \tag{2.52}$$

Assuming that the action of simple currents on fields is also linear and using the notation $g^{-1}\phi g\equiv\mathcal{R}(g)\phi$ it can be written as

$$\begin{aligned} \mathcal{R}(g_1^{-1})\phi_1 &= \mathcal{R}(P)\mathcal{R}(g_2^{-1})\phi_2, \\ \mathcal{R}(g_2^{-1})\phi_2 &= \mathcal{R}(P)\mathcal{R}(g_1^{-1})\phi_1. \end{aligned} \tag{2.53}$$

It follows that the field ϕ defined as in (2.51) satisfies the twisted periodicity, as claimed above:

$$\phi(\sigma) \;=\; \mathcal{R}(g_1g_2)\phi(\sigma-2\pi) \;=\; (g_1g_2)^{-1}\phi(\sigma-2\pi)g_1g_2. \tag{2.54}$$

Second, the overlaps of diagonal boundary states in $(h\otimes h^{-1})$-twisted sector correspond to path integral over fields of $\mathcal{X}^2$ on a cylinder with the twisted periodicity along t,

$$\phi_1(\sigma,t) \;=\; h\phi_1(\sigma,t-2\pi l)h^{-1}, \qquad \phi_2(\sigma,t) \;=\; h^{-1}\phi_2(\sigma,t-2\pi l)h. \tag{2.55}$$

In terms of the field ϕ this is simply

$$\phi(\sigma,t) \;=\; h\phi(\sigma,t-2\pi l)h^{-1}. \tag{2.56}$$

From these two observations it follows that the diagonal branes of $\mathcal{X}^2$ sitting in twisted sectors satisfy the formula

$${}^{h\otimes h^{-1}}\langle\mathcal{B}_{\rm diag}|e^{-\pi H/l}(g_1\otimes g_2)|\mathcal{B}_{\rm diag}\rangle^{h\otimes h^{-1}} \;=\; {\rm Tr}^{\mathcal{X}}_{g_1^{-1}g_2^{-1}}[he^{-2\pi Hl}]. \tag{2.57}$$

By comparing this with (2.48), one can check that the RCFT construction gives the diagonal branes with the correct property.

We have seen in the previous subsection that the PSS prescription allows to construct crosscaps corresponding to different parity symmetry. The fundamental diagonal brane we have studied above should be associated to the fundamental parity P corresponding to the crosscap $|\mathcal{C}_0\rangle$. The diagonal branes corresponding to other parities are obtained by a similar argument as was given above. For each representation I of $\mathcal{A}$ satisfying the fusion rule $I\times\bar{I}\longmapsto{\rm id}$, there is a boundary state $|\mathcal{B}_I^{(12)}\rangle$ in $\mathcal{X}^2$

$$|\mathcal{B}_I^{(12)}\rangle \;=\; \sum_i\frac{S_{Ii}}{S_{0i}}R^{(12)}\,|\mathcal{B};i,i\rangle\!\rangle. \tag{2.58}$$

The fields of the two copies of $\mathcal{X}$ are glued via the parity P_I. The corresponding diagonal branes in the orbifold are given by

$$|\mathcal{B}_I^{(12),\rho}\rangle^{\rm orb} \;=\; \frac{1}{\sqrt{|\mathcal{G}|}}\sum_{g,h\in G}(g\otimes 1)|\mathcal{B}_I^{(12)}\rangle^{h\otimes h^{-1}}\rho(h), \tag{2.59}$$

where $\rho(h)$ is a character of (the double cover of) G, and the boundary states in twisted sectors are defined as

$$|\mathcal{B}_I^{(12)}\rangle^{h\otimes h^{-1}} \;=\; \sum_i \frac{S_{Ii}}{S_{0i}} R^{(12)}\,|\mathcal{B};h(i),i\rangle\!\rangle e^{i\pi Q_h(I)}, \tag{2.60}$$

where the last factor is added so that $(g_1\otimes g_2)|\mathcal{B}_I^{(12)}\rangle^{h\otimes h^{-1}} = |\mathcal{B}_{g_1g_2I}^{(12)}\rangle^{h\otimes h^{-1}}$ is satisfied. Note that (2.60) in general has doubled periodicity as a function of h, so $\rho(h)$ in (2.59) should be chosen so that the summand of the right hand side is single valued.

2.4 Permutation crosscaps in orbifolds

Let us next construct permutation crosscaps in orbifolds.[2] For an involutive permutation $\pi \in S_N$ and a π-invariant parity P_I of $\mathcal{X}^N$, PSS's construction gives us the crosscap state in $\mathcal{X}^N$ corresponding to the parity $P_I\pi$. To obtain crosscap states in the orbifold $\mathcal{X}^N/\mathcal{G}$, one needs crosscaps corresponding to the parities $gP_I\pi$ ($g\in\mathcal{G}$) which map the states of $\mathcal{X}^N$ as follows:

$$gP_I\pi:\; a_1\otimes\cdots\otimes a_N \;\to\; (g_1P_{I_1}a_{\pi(1)})\otimes\cdots\otimes(g_NP_{I_N}a_{\pi(N)}). \tag{2.61}$$

The permutation crosscaps in $\mathcal{X}^N/\mathcal{G}$ are sums over those in $\mathcal{X}^N$,

$$|P_I^{\pi,\epsilon}\rangle^{\mathcal{X}^N/\mathcal{G}} \;=\; \frac{1}{\sqrt{|\mathcal{G}|}}\sum_{g\in\mathcal{G}} |gP_I\pi\rangle^{\mathcal{X}^N}\epsilon(g), \tag{2.62}$$

dressed by a character $\epsilon(g)$ of (the double cover of) $\mathcal{G}$ satisfying suitable periodicity conditions. The $\mathcal{G}$-invariance of the crosscap state requires $\epsilon(gg_\pi)=1$ for all $g\in\mathcal{G}$, but it does not necessarily require that ϵ be $\mathbb{Z}_2$-valued. Note also that, for the equation (2.62) to define an involutive parity in the orbifold, P_I actually does not have to be involutive; it only has to square to an element of $\mathcal{G}$.

The π-permuted crosscap states should factorize into pieces representing the cycles of π,

$$|gP_I\pi\rangle \;=\; \otimes_{c=1}^{[\pi]}|g_cP_{I_c}\pi_c\rangle, \tag{2.63}$$

where all the cycles of π are of length one or two because π is involutive. For cycles of length one we have seen the correspondence (2.35), so it remains to construct the crosscaps $|gP_I\pi\rangle$ for the cyclic permutation of length two, $\pi=(1\,2)$.

We focus first on the crosscap $|gP\pi\rangle$ corresponding to the fundamental PSS parity P. The overlaps of two permutation crosscaps $\langle gP\pi|$ and $|\tilde{g}P\pi\rangle$ correspond to the theory $\mathcal{X}^2$ on a space $(\sigma\in[0,\pi], t\sim t+4\pi l)$ with boundary conditions

$$\begin{aligned}\phi_1(0,t) &= \mathcal{R}(P)\mathcal{R}(g_2^{-1})\phi_2(0,t-2\pi l),\\ \phi_2(0,t) &= \mathcal{R}(P)\mathcal{R}(g_1^{-1})\phi_1(0,t-2\pi l),\\ \phi_1(\pi,t) &= \mathcal{R}(P)\mathcal{R}(\tilde{g}_2^{-1})\phi_2(\pi,t-2\pi l),\\ \phi_2(\pi,t) &= \mathcal{R}(P)\mathcal{R}(\tilde{g}_1^{-1})\phi_1(\pi,t-2\pi l).\end{aligned} \tag{2.64}$$

[2]The outline of the argument in this subsection was suggested to us by K. Hori.

As states in the Hilbert space of the theory $\mathcal{X}^2$, the crosscap states $\langle gP\pi|$ and $|\tilde{g}P\pi\rangle$ belong to the sector twisted by $gg_\pi^{-1} = (g_1g_2^{-1} \otimes g_2g_1^{-1})$ and $\tilde{g}\tilde{g}_\pi^{-1}$, respectively. Therefore, $g_1g_2^{-1} \equiv \tilde{g}_1\tilde{g}_2^{-1}$ for pairs of crosscaps with nonzero overlaps. By arguing in a similar way to the construction of diagonal branes, one finds that the theory $\mathcal{X}^2$ with boundary conditions (2.64) is equivalent to the theory $\mathcal{X}$ on torus $(\sigma \sim \sigma + 2\pi, t \sim t + 4\pi l)$ with periodicity,

$$\phi(\sigma,t) \;=\; g_1\tilde{g}_1^{-1}\phi(\sigma-2\pi,t)g_1^{-1}\tilde{g}_1 \;=\; g_1g_2^{-1}\phi(\sigma,t-4\pi l)g_2g_1^{-1}. \tag{2.65}$$

The overlaps of permutation crosscaps thus gives the torus partition function of the theory $\mathcal{X}$,

$$\langle gP\pi|e^{-\pi H_c/l}|\tilde{g}P\pi\rangle = \mathrm{Tr}^{\mathcal{X}}_{g_1\tilde{g}_1^{-1}}[g_1g_2^{-1}e^{-4\pi H_c l}]. \tag{2.66}$$

We need the formula for permutation crosscaps expressed in terms of Ishibashi states in twisted sectors,

$$|gP\pi\rangle \;=\; \sum_i X_i(g_1,g_2)\,|\mathcal{C}^\pi; g_1 i, g_2 i\rangle\!\rangle. \tag{2.67}$$

We determine it by requiring that it has the following overlap with the fundamental diagonal brane,

$$\langle \mathcal{B}_{\mathrm{diag}}|e^{-\pi H_c/2l}|gP\pi\rangle \;=\; \mathrm{Tr}^{\mathcal{X}}_{\mathcal{H}_{g_2^{-1}g_1^{-1}}}[g_1e^{-2\pi lH-i\pi P}] \;=\; \mathrm{Tr}^{\mathcal{X}}_{\mathcal{H}_{g_2^{-1}g_1^{-1}}}[g_2^{-1}e^{-2\pi lH+i\pi P}], \tag{2.68}$$

where one should recall

$$H = L_0 + \bar{L}_0 - \frac{c}{12}, \quad P = L_0 - \bar{L}_0.$$

To understand this condition, let us consider the theory $\mathcal{X}^2$ on a strip $(0 \le \sigma \le \pi)$ bounded by the diagonal brane $\langle\mathcal{B}_{\mathrm{diag}}|$ and its image under the parity $gP\pi$. The partition function on the Möbius strip is calculated by the path integral of fields $\phi_{1,2}$ of $\mathcal{X}^2$ with the following boundary condition at $\sigma = 0$,

$$\begin{aligned}\phi_1(0,t) &= \mathcal{R}(P)\phi_2(0,t),\\ \phi_2(0,t) &= \mathcal{R}(P)\phi_1(0,t),\end{aligned} \tag{2.69}$$

and the periodicity along the t-direction,

$$\begin{aligned}\phi_1(\sigma,t) &= \mathcal{R}(P)\mathcal{R}(g_2^{-1})\phi_2(\pi-\sigma,t-2\pi l),\\ \phi_2(\sigma,t) &= \mathcal{R}(P)\mathcal{R}(g_1^{-1})\phi_1(\pi-\sigma,t-2\pi l).\end{aligned} \tag{2.70}$$

It follows that the boundary condition at $\sigma = \pi$ has to be that of $(g_1 \otimes g_2)|\mathcal{B}_{\mathrm{diag}}\rangle$, (2.53). Thus the theory $\mathcal{X}^2$ on Möbius strip is equivalent to the theory $\mathcal{X}$ on the torus, with the field ϕ satisfying the periodicity along the spatial direction (2.54), and the time direction

$$\phi(\sigma,t) \;=\; g_2^{-1}\phi(\sigma-\pi,t-2\pi l)g_2 \;=\; g_1\phi(\sigma+\pi,t-2\pi l)g_1^{-1}, \tag{2.71}$$

hence the requirement (2.68). We solve it and find

$$|gP\pi\rangle = \sum_i |\mathcal{C}^\pi; g_1 i, g_2 i\rangle\rangle \exp \pi i \left(2q(g_1, g_1 g_2) + 2h_{g_1} + 2h_i - h_{g_1 i} - h_{g_2 i}\right)$$
$$= \sum_i |\mathcal{C}^\pi; g_1 i, g_2 i\rangle\rangle \sigma_{g_1 i}\sigma_{g_2 i} \exp \pi i \left\{q(g_1 g_2, g_1 g_2) + Q_{g_1 g_2}(i)\right\}. \quad (2.72)$$

The expression for more general permutation crosscaps $|gP_I\pi\rangle$ can be found by studying its overlap with the diagonal brane $|\mathcal{B}^\pi_I\rangle$. Our final result reads

$$|gP_I\pi\rangle = \sum_i \frac{S_{Ii}}{S_{0i}} |\mathcal{C}^\pi; g_1 i, g_2 i\rangle\rangle \sigma_{g_1 i}\sigma_{g_2 i} \exp \pi i \left\{q(g_1 g_2, g_1 g_2) + Q_{g_1 g_2}(i) + Q_{g_1 g_2}(I)\right\}. \quad (2.73)$$

Note that this crosscap has the same periodicity as that of $\sigma_{g_1 g_2(I)}\sigma_{g_1 g_2}$.

2.5 Parity action on D-branes

The action of parity $P_I\pi$ on branes in $\mathcal{X}^N$ is read off from the relation

$$\langle \mathcal{B}| q_t^H |P_I\pi\rangle = \langle P_I\pi| q_t^H |\mathcal{B}'\rangle. \quad (2.74)$$

When $|\mathcal{B}\rangle$ is a σ-permuted brane gluing the a-th holomorphic sector with the $\sigma(a)$-th antiholomorphic sector, its parity image $|\mathcal{B}'\rangle$ should glue the $\pi(a)$-th antiholomorphic sector with the $\pi\sigma(a)$-th holomorphic sector. So $|\mathcal{B}'\rangle$ has to be a $\sigma' = \pi\sigma^{-1}\pi$-permuted brane. One then finds, using

$$\langle\langle \mathcal{B}; i_1 \cdots i_N | R^{\sigma^{-1}} q_t^H R^\pi |\mathcal{C}; j_1 \cdots j_N\rangle\rangle = \langle\langle \mathcal{C}; j_1 \cdots j_N | R^{\pi^{-1}} q_t^H R^{\sigma'} |\mathcal{B}; \tilde{\imath}_1 \cdots \tilde{\imath}_N\rangle\rangle, \quad (2.75)$$

where $\tilde{\imath}_a = i_{\sigma^{-1}\pi(a)}$, that the parity acts on boundary states as follows:

$$\langle \mathcal{B}^\sigma_{\mathbf{J}}| q_t^H |P_I\pi\rangle = \langle P_{\bar{I}}\pi| q_t^H |\mathcal{B}^{\sigma'}_{\bar{\mathbf{J}}}\rangle = \langle P_I\pi| q_t^H \omega |\mathcal{B}^{\sigma'}_{\bar{\mathbf{J}}}\rangle, \quad (2.76)$$

where ω is a simple current satisfying $\omega_\pi \omega \bar{I} = I$. Although there may be several ω's satisfying this, there must be a unique ω that determines the action of parity $P_I\pi$ on D-branes. For example, for the permutation crosscaps $|gP\pi\rangle$ made from the fundamental parity P and $g = (g_1, \cdots, g_N)$, one finds both from the Möbius strip amplitudes of RCFT and from a formal field theory argument that

$$\langle \mathcal{B}^\sigma_{\mathbf{J}}| q_t^H |P\pi\rangle = \langle P\pi| q_t^H |\mathcal{B}^{\sigma'}_{\bar{\mathbf{J}}}\rangle \Longrightarrow \langle \mathcal{B}^\sigma_{\mathbf{J}}| q_t^H |gP\pi\rangle = \langle gP\pi| q_t^H g |\mathcal{B}^{\sigma'}_{\bar{\mathbf{J}}}\rangle. \quad (2.77)$$

Note here that the labels $\mathbf{J}, \bar{\mathbf{J}}$ denote the sets of representations $\{J_c\}, \{\bar{J}_c\}$ $(c = 1, \cdots, [\sigma])$. J_c and $\bar{J}_c$ are for the c-th cycle of σ and σ', which are conjugate to each other thanks to π being involutive.

By a similar argument one can derive the action of parity $P_I^{\pi,\epsilon}$ (2.62) on branes in orbifold $\mathcal{X}^N/\mathcal{G}$. We notice that (2.75) relates the bra Ishibashi states in the h-twisted sector to the ket Ishibashi states in h_π^{-1}-twisted sector. The Möbius strip amplitude of the orbifold theory,

$$\langle \mathcal{B}^{\sigma,\rho}_{\mathbf{J}}| q_t^H |P_I^{\pi,\epsilon}\rangle \sim \sum_{g,h} \rho^*(h) \cdot {}^h\langle \mathcal{B}^\sigma_{\mathbf{J}}| q_t^H |gP_I\pi\rangle^h \epsilon(g) \cdot \delta_{h, g g_\pi^{-1}}, \quad (2.78)$$

allows us to read off the parity action on boundary states:

$$P_I^{\pi,\epsilon} \;:\; |\mathcal{B}_{\mathbf{J}}^{\sigma,\rho}\rangle \;\longmapsto\; \epsilon(\omega)|\mathcal{B}_{\mathbf{J}}^{\sigma',\rho'}\rangle; \qquad \bar{I} = \omega I, \quad \rho'(h_\pi) = \rho(h)\epsilon(h)^{-1}. \tag{2.79}$$

The transformation law of $\rho(h)$ means that the parity $P_I^{\pi,\epsilon}$ maps states in h-twisted sector to those in h_π-twisted sector after multiplying $\epsilon(h)^{-1}$, a fact which follows also from the construction of permutation parities in orbifold.

The above expression is still somewhat ambiguous because of the subtlety mentioned after (2.42): we need to specify the first element for each cycle of σ to define Ishibashi states in twisted sectors unambiguously. If $\sigma = (a_{_1} \cdots a_{_N})$ is a single cycle and $\pi\sigma^{-1}\pi = (a'_{_1} \cdots a'_{_N})$, then we have to put $i_{a_N} = \tilde{i}_{a'_N}$ in (2.75) and get

$$\pi\sigma^{-1}\pi \;=\; (a'_1 \cdots a'_N) \;=\; (\pi(a_N)\cdots\pi(a_1)). \tag{2.80}$$

2.5.1 Parity invariant D-branes

As a future reference, we study the condition of parity-invariance for permutation branes in more detail. Here we give the condition on the pair (π, σ) in order for the σ-permuted brane to be invariant under π-permuted orientifold.

Condition for Parity Invariant Branes (PIB) 1. *Any pair of permutation π, σ satisfying $\sigma = \pi\sigma^{-1}\pi, \pi^2 = \mathrm{id}$ can be decomposed into the following blocks,*

$$\begin{array}{lll}
(1)\ \sigma = (a_1a_2\cdots a_{2n+1}), & \pi = (a_1a_{2n+1})(a_2a_{2n})\cdots(a_na_{n+2}), \\
(2)\ \sigma = (a_1a_2\cdots a_{2n}), & \pi = (a_2a_{2n+1})(a_2a_{2n})\cdots(a_na_{n+2}), \\
(3)\ \sigma = (a_1a_2\cdots a_{2n}), & \pi = \quad (a_1a_{2n})(a_2a_{2n})\cdots(a_na_{n+1}), \\
(4)\ \sigma = (a_1\cdots a_n)(a'_1\cdots a'_n), & \pi = \quad (a_1a'_n)(a_2a'_{n-1})\cdots(a_na'_1).
\end{array}$$

The simplest block $\sigma = \pi = \mathrm{id} \in S_1$ is a special case of the first type, and $\sigma = (a_1a_2)$, $\pi = \mathrm{id} \in S_2$ is the simplest example of the second type. The permutation $\sigma^{-1}\pi$ or its inverse appear in Möbius strip amplitudes as explained in (2.9). Note $\sigma^{-1}\pi$ always squares to identity because of $\sigma = \pi\sigma^{-1}\pi$, so it consists of cycles of lengths one or two only.

In general, the spectrum of open string between identical D-branes contains an identity representation. The Möbius strip amplitude for parity-invariant boundary states, when written in the loop channel, should therefore contain an identity character. To check this explicitly, we need to show

$$\langle\mathcal{C}^\pi|e^{-\pi H_c/4l}|\mathcal{B}^\sigma\rangle \;\sim\; q^{-\frac{Nc_\mathcal{X}}{24}} + \cdots. \qquad (q \equiv e^{-2\pi l}) \tag{2.81}$$

Here $-\frac{Nc_\mathcal{X}}{24}$ is the energy for the $\mathrm{SL}(2,\mathbb{R})$-invariant ground state. The amplitude can be written in the tree channel as a sum of the following products of characters,

$$\prod_{\|\tilde{\sigma}_a\|\ \text{even}} \chi_{j_a}(\tfrac{i\|\tilde{\sigma}_a\|}{4l}) \prod_{\|\tilde{\sigma}_b\|\ \text{odd}} \hat{\chi}_{j_b}(\tfrac{i\|\tilde{\sigma}_b\|}{4l}) \qquad (\tilde{\sigma} \equiv \sigma^{-1}\circ\pi),$$

where one should recall that each character χ_{j_a} or $\widehat{\chi}_{j_b}$ corresponds to a cycle of $\tilde{\sigma}$ of even or odd length. One can read off the energy E_0 of the ground state of the open string Hilbert space by modular transform,

$$E_0 \;=\; -\sum_{\|\tilde{\sigma}_a\|\ \text{even}} \frac{c_{\mathcal{X}}}{6\|\tilde{\sigma}_a\|} - \sum_{\|\tilde{\sigma}_b\|\ \text{odd}} \frac{c_{\mathcal{X}}}{24\|\tilde{\sigma}_a\|}.$$

This saturates the lower bound $-Nc_{\mathcal{X}}/24$ iff all the cycles of $\tilde{\sigma}$ have length one or two. The four types of parity-invariant boundary states listed above all satisfy this condition.

3. Dirac Fermion and the affine $\mathrm{U}(1)_2$ model

In this section we illustrate the construction of permutation branes and orientifolds in the theory of d Dirac fermions $\psi^{\pm,a}$. It is pretty obvious how to construct the boundary or crosscap states satisfying

$$\begin{aligned}(\tilde{\psi}_n^{\pm,\pi(a)} + i\eta\psi_{-n}^{\pm,a})|\mathcal{B}^\pi\rangle_{\mathrm{Y}\eta} &= 0,\\ (\tilde{\psi}_n^{\pm,\pi(a)} + i\eta e^{i\pi n}\psi_{-n}^{\pm,a})|\mathcal{C}^\pi\rangle_{\mathrm{Y}\eta} &= 0,\end{aligned} \qquad (\mathrm{Y} = \text{NSNS or RR}\ ;\ \eta = \pm) \tag{3.1}$$

as Bogolioubov transforms of the vacuum following [40, 41]. On the other hand, one can construct the same states from the boundary or crosscap states in the affine $\mathrm{U}(1)_2^d$ model by a suitable $(\mathbb{Z}_2)^d$ orbifold. Since the affine $\mathrm{U}(1)_2^d$ theory or its orbifold is purely bosonic, one must assign Grassmann parity to the operators and states in a suitable manner to reproduce the properties of fermions correctly, as we will discuss here in detail. The result obtained here also has a direct application to Gepner's construction of superstring theories, where supersymmetric worldsheet theories are constructed from purely bosonic RCFTs by the same orbifold.

The affine $\mathrm{U}(1)_k$ symmetry is generated by the current $J = i\sqrt{2k}\partial X$ augmented by spectral flow operators $e^{\pm i\sqrt{2k}X}$, where X is a canonically normalized chiral scalar field. There are $2k$ highest weight representations labelled by a mod $2k$ integer n corresponding to the collection of operators $e^{iqX/\sqrt{2k}}$ ($q = n \bmod 2k$) and their descendants. The $\mathrm{U}(1)$ charge and conformal weight of the operator $e^{iqX/\sqrt{2k}}$ are $(J_0, L_0) = (q, \frac{q^2}{4k})$. The model at level $k = 2$ has four representations labelled by an integer $s \sim s + 4$. We denote by ψ the simple current satisfying the fusion rule $\psi(s) = s + 2$.

The affine $\mathrm{U}(1)_2$ theory is related to the theory of a Dirac fermion by the $\mathbb{Z}_2$-orbifolding. This fact can be seen from the relation of characters: from the characters of the affine $\mathrm{U}(1)_2$ algebra,

$$\chi_s(\tau,\nu) \;\equiv\; \mathrm{Tr}_{[s]} q^{L_0-1/24} z^{J_0/2} \;=\; \eta(\tau)^{-1} \sum_{l\in\mathbb{Z}+s/4} q^{2l^2} z^{2l}, \quad (q = e^{2\pi i\tau},\ z = e^{2\pi i\nu}) \tag{3.2}$$

one can construct characters of Dirac fermion model,

$$\begin{aligned}\chi_0 \pm \chi_2 \;&=\; \chi^{\mathrm{NS}\pm}(\tau,\nu) \;=\; q^{-\frac{1}{24}} \prod_{m\geq 1} (1 \pm zq^{m-\frac{1}{2}})(1 \pm z^{-1}q^{m-\frac{1}{2}}),\\ \chi_1 \pm \chi_{-1} \;&=\; \chi^{\mathrm{R}\pm}(\tau,\nu) \;=\; q^{\frac{1}{12}}(z^{\frac{1}{2}} \pm z^{-\frac{1}{2}}) \prod_{m\geq 1} (1 \pm zq^m)(1 \pm z^{-1}q^m).\end{aligned} \tag{3.3}$$

The theory of d Dirac fermions is obtained from the affine $\mathrm{U}(1)_2^d$ model by orbifolding by $\Gamma_{\mathrm{GSO}} \equiv (\mathbb{Z}_2)^d$ generated by the simple currents ψ_a, with the choice $q \equiv 0$. The choice $q \equiv 0$ does not give a modular invariant torus partition function because it does not satisfy (2.19), but the modular invariance is recovered by summing over four spin structures. In RCFT terms, different spin structures arise from (i) restricting to states for which the eigenvalues of all ψ_a are aligned, i.e. $\psi_a = 1(\forall a)$ for NSNS sector or (-1) for RR sector, and (ii) summing over twisted sectors with trivial weight or weighted by a nontrivial character $\epsilon : \Gamma_{\mathrm{GSO}} \mapsto \mathbb{Z}_2$ such that $\epsilon(\psi_a) = -1(\forall a)$. It is easy to see that the orbifold by Γ_{GSO} and summing over spin structures gives the same torus partition function as the orbifold by a group $\widetilde{\Gamma}_{\mathrm{GSO}} = (\mathbb{Z}_2)^{d-1}$ of *even* monomials of ψ_a. The orbifold group $\widetilde{\Gamma}_{\mathrm{GSO}}$ is used in Gepner's original construction of superstring models [1].

3.1 D-branes

The quartet of boundary states in Dirac fermion theory should be obtained from those in affine $\mathrm{U}(1)_2$ theory by orbifolding,

$$\begin{aligned}
|\mathcal{B}\rangle_{\mathrm{NSNS}+} &= |\mathcal{B};0\rangle\!\rangle^{\mathrm{U}(1)} + |\mathcal{B};2\rangle\!\rangle^{\mathrm{U}(1)} &&= \tfrac{1}{\sqrt{2}}(|\mathcal{B}_0\rangle^{\mathrm{U}(1)} + |\mathcal{B}_2\rangle^{\mathrm{U}(1)}),\\
|\mathcal{B}\rangle_{\mathrm{NSNS}-} &= |\mathcal{B};0\rangle\!\rangle^{\mathrm{U}(1)} - |\mathcal{B};2\rangle\!\rangle^{\mathrm{U}(1)} &&= \tfrac{1}{\sqrt{2}}(|\mathcal{B}_1\rangle^{\mathrm{U}(1)} + |\mathcal{B}_{-1}\rangle^{\mathrm{U}(1)}),\\
|\mathcal{B}\rangle_{\mathrm{RR}+} &= |\mathcal{B};1\rangle\!\rangle^{\mathrm{U}(1)} + |\mathcal{B};-1\rangle\!\rangle^{\mathrm{U}(1)} &&= \tfrac{1}{\sqrt{2}}(|\mathcal{B}_0\rangle^{\mathrm{U}(1)} - |\mathcal{B}_2\rangle^{\mathrm{U}(1)}),\\
|\mathcal{B}\rangle_{\mathrm{RR}-} &= -i\,|\mathcal{B};1\rangle\!\rangle^{\mathrm{U}(1)} + i\,|\mathcal{B};-1\rangle\!\rangle^{\mathrm{U}(1)} &&= \tfrac{1}{\sqrt{2}}(|\mathcal{B}_1\rangle^{\mathrm{U}(1)} - |\mathcal{B}_{-1}\rangle^{\mathrm{U}(1)}).
\end{aligned} \tag{3.4}$$

Here the Ishibashi and Cardy states of the affine $\mathrm{U}(1)_2$ theory are related by the standard formula

$$|\mathcal{B}_S\rangle^{\mathrm{U}(1)} \;=\; \sum_s \frac{S_{Ss}}{\sqrt{S_{0s}}}\,|\mathcal{B};s\rangle\!\rangle^{\mathrm{U}(1)}, \qquad S_{Ss} = \frac{1}{2}e^{-i\pi Ss/2}. \tag{3.5}$$

We would like to make sure that the boundary states (3.4) constructed from those in $\mathrm{U}(1)_2$ theory indeed satisfy the boundary conditions on Dirac fermions $\psi^{\pm}(z), \tilde{\psi}^{\pm}(\bar{z})$,

$$(\tilde{\psi}^{\pm}_n + i\eta\psi^{\pm}_{-n})|\mathcal{B}\rangle_{\mathrm{Y},\eta} \;=\; 0. \tag{3.6}$$

We first notice that $\psi^{\pm} = e^{\pm iX}$ correspond to nothing but the simple current ψ in the affine $\mathrm{U}(1)_2$ theory. It induces invertible maps from $\mathcal{V}_s$ to $\mathcal{V}_{s+2}$ that square to the identity. There are infinitely many such maps; for example the multiplication of $(\psi^+_r + \psi^-_{-r})$ is easily seen to square to unity. Pick an arbitrary such map and denote it by Ψ. On closed string Hilbert space, one can thus consider operators $\Psi, \widetilde{\Psi}$ acting on the right and left-moving sectors respectively. For a suitably chosen basis of orthonormal states, they satisfy

$$\begin{aligned}
\Psi(|s,M\rangle \otimes |\tilde{s},\tilde{M}\rangle) &= |s+2,M\rangle \otimes |\tilde{s},\tilde{M}\rangle,\\
\widetilde{\Psi}(|s,M\rangle \otimes |\tilde{s},\tilde{M}\rangle) &= |s,M\rangle \otimes |\tilde{s}+2,\tilde{M}\rangle(-i)(-)^{\frac{s-\tilde{s}}{2}}.
\end{aligned} \tag{3.7}$$

where $|s,M\rangle$ denotes the M-th state in the representation $[s]$ of affine $\mathrm{U}(1)_2$. The phase factor in the second equation was chosen so that the relations $\Psi^2 = \widetilde{\Psi}^2 = \mathrm{id}$, $\Psi\widetilde{\Psi} + \widetilde{\Psi}\Psi = 0$ hold. The boundary states defined in (3.4) then satisfy

$$(\widetilde{\Psi} \pm i\Psi)|\mathcal{B}\rangle_{\mathrm{NSNS}\pm} \;=\; 0, \qquad (\widetilde{\Psi} \mp i\Psi)|\mathcal{B}\rangle_{\mathrm{RR}\pm} \;=\; 0, \tag{3.8}$$

for any choice of $(\Psi, \widetilde{\Psi})$ corresponding to the simple current ψ. We regard this as corresponding to the boundary condition on fermions (3.6).

Let us try to extend the argument to general permutation branes in the theory of d Dirac fermions. We wish to find a quartet of boundary states in the orbifold $\mathrm{U}(1)_2^{\otimes d}/(\mathbb{Z}_2)^d$ satisfying the boundary condition

$$(\widetilde{\Psi}^{\pi(a)} \pm i\Psi^a)|\mathcal{B}\rangle_{\text{NSNS}\pm} \;=\; 0, \qquad (\widetilde{\Psi}^{\pi(a)} \mp i\Psi^a)|\mathcal{B}\rangle_{\text{RR}\pm} \;=\; 0, \tag{3.9}$$

for any map Ψ associated to the simple current in $\mathrm{U}(1)_2$ model. The operators $\Psi^a, \widetilde{\Psi}^a$ act on the states of the a-th $\mathrm{U}(1)_2$ theory as (3.7), but we also need to determine how to pass them through the states of the first $(a-1)$ theories. It should be determined in such a way that the maps Ψ_a and $\widetilde{\Psi}_a$ anticommute with one another.

Hereafter we work with the assignment that the state $|s, M\rangle$ is Grassmann even when $s = 0$ or 1, and otherwise Grassmann odd. This Grassmann parity has to be taken care of when the states are permuted by operations such as R^π (2.5) in constructing permutation branes. In the following discussions, we denote by R^π the permutation operation with Grassmann parity taken into account, and by $R^\pi_\circ$ the one neglecting the Grassmann parity. The two operations therefore differ by $\pm$ signs when action on general states or operators.

To understand how the effect of Grassmann parity enters into the definition of boundary states, let us consider the simplest permutation brane in two Dirac fermion theory. The boundary states are sums of states in the untwisted and twisted sectors. The untwisted part is given by

$$\begin{aligned}
\frac{1}{2}\left(|\mathcal{B}_S^{(12)}\rangle^{\mathrm{U}(1)^2} + |\mathcal{B}_{S+2}^{(12)}\rangle^{\mathrm{U}(1)^2}\right) &= \sum_{s=0,2} e^{-\frac{\pi i S s}{2}} R_\circ^{(12)}\, |\mathcal{B}; s, s\rangle\!\rangle = \sum_{s=0,2} e^{-\frac{\pi i S s}{2}} (-)^{\frac{s}{2}} R^{(12)}\, |\mathcal{B}; s, s\rangle\!\rangle, \\
\frac{1}{2}\left(|\mathcal{B}_S^{(12)}\rangle^{\mathrm{U}(1)^2} - |\mathcal{B}_{S+2}^{(12)}\rangle^{\mathrm{U}(1)^2}\right) &= \sum_{s=\pm 1} e^{-\frac{\pi i S s}{2}} R_\circ^{(12)}\, |\mathcal{B}; s, s\rangle\!\rangle = \sum_{s=\pm 1} e^{-\frac{\pi i S s}{2}} (-)^{\frac{s+1}{2}} R^{(12)}\, |\mathcal{B}; s, s\rangle\!\rangle.
\end{aligned} \tag{3.10}$$

These define two NSNS and two RR boundary states. The sign factors $(-)^{\frac{s}{2}}$ or $(-)^{\frac{s+1}{2}}$ arise from exchanging states by $R^{(12)}$. The above states with $S = 1$ can satisfy the boundary condition (3.9) when suitable states in the twisted sector are added, whereas the states with $S = 0$ cannot. The quartet of permutation boundary states is thus given by

$$\begin{aligned}
|\mathcal{B}^{(12)}\rangle_{\text{NSNS}\pm} &= \frac{1}{2}\sum_{h\in\mathcal{H}} \rho_\pm(h)\left(|\mathcal{B}_1^{(12)}\rangle^h + |\mathcal{B}_{-1}^{(12)}\rangle^h\right), \\
|\mathcal{B}^{(12)}\rangle_{\text{RR}\pm} &= \pm\frac{1}{2}\sum_{h\in\mathcal{H}} \rho_\mp(h)\left(|\mathcal{B}_1^{(12)}\rangle^h - |\mathcal{B}_{-1}^{(12)}\rangle^h\right),
\end{aligned} \tag{3.11}$$

where $\mathcal{H} = \mathbb{Z}_2$ is the stabilizer group generated by $\psi_1\psi_2$, and ρ_+ (ρ_-) is the trivial (resp. nontrivial) character of $\mathcal{H}$. They can actually be rewritten in a simple form,

$$\begin{aligned}
|\mathcal{B}^{(12)}\rangle_{\text{NSNS}\pm} &= R^{(12)}(|\mathcal{B}\rangle_{\text{NSNS}\pm})^{\otimes 2}, \\
|\mathcal{B}^{(12)}\rangle_{\text{RR}\pm} &= iR^{(12)}(|\mathcal{B}\rangle_{\text{RR}\pm})^{\otimes 2}.
\end{aligned} \tag{3.12}$$

The overlap of the states $|\mathcal{B}^{(12)}\rangle_{\rm NSNS\pm}$ with the ordinary branes $|\mathcal{B}^{(1)(2)}\rangle_{\rm NSNS\pm}$ is always given by the character of Ramond representation in the loop channel,

$$ {}_{\rm NSNS,\epsilon}\langle \mathcal{B}^{(1)(2)}|e^{-\pi H/l}|\mathcal{B}^{(12)}\rangle_{\rm NSNS,\epsilon'} \;=\; \chi^{\rm NS-}(2i/l) \;=\; \chi^{\rm R+}(il/2). \tag{3.13} $$

Here the characters are those given in (3.3) with ν set to zero. This is easily seen to be consistent with the boundary condition on supercurrent.

The construction of branes corresponding to cyclic permutations of lengths $N \geq 3$ goes in a similar way. The boundary states are sums of the states $|\mathcal{B}^{\pi}_{S}\rangle^{h}, |\mathcal{B}^{\pi}_{S+2}\rangle^{h}$ over the twists $h \in (\mathbb{Z}_2)^{N-1}$ with suitable weights. There are two distinguished weights for which the boundary conditions on fermions are all appropriately aligned. It also turns out that one has to choose $S = 1$ for all spin structures when the cycle has even length.

3.2 Orientifolds

We start with constructing the orientifold of a Dirac fermion theory via $\mathbb{Z}_2$ orbifold of $\mathrm{U}(1)_2$ theory. Since the choice $q \equiv 0$ is somewhat unnatural, our starting formula is (2.32). Defining the basic parity P by the action (1.5) on Dirac fermions, one can consider the quartet of parity symmetries $\epsilon^{F_R}\tilde{\epsilon}^{F_L}P$ defined by the action on fields on a strip,

$$ \begin{aligned} (\epsilon^{F_R}\tilde{\epsilon}^{F_L}P)\psi^{\pm}(\sigma,t)(\epsilon^{F_R}\tilde{\epsilon}^{F_L}P)^{-1} &= \tilde{\epsilon}e^{-i\pi/2}\tilde{\psi}^{\pm}(\pi-\sigma,t), \\ (\epsilon^{F_R}\tilde{\epsilon}^{F_L}P)\tilde{\psi}^{\pm}(\sigma,t)(\epsilon^{F_R}\tilde{\epsilon}^{F_L}P)^{-1} &= \epsilon e^{+i\pi/2}\psi^{\pm}(\pi-\sigma,t). \end{aligned} \tag{3.14} $$

The quartet of crosscap states is constructed by applying the formula (2.32),

$$ \begin{aligned} |(-)^{F_L}P\rangle &\equiv |\mathcal{C}\rangle_{\rm NSNS+} &&= \tfrac{1}{\sqrt{2}}(|\mathcal{C}_0\rangle^{\mathrm{U}(1)} - i|\mathcal{C}_2\rangle^{\mathrm{U}(1)})e^{i\beta}, \\ |(-)^{F_R}P\rangle &\equiv |\mathcal{C}\rangle_{\rm NSNS-} &&= \tfrac{1}{\sqrt{2}}(|\mathcal{C}_0\rangle^{\mathrm{U}(1)} + i|\mathcal{C}_2\rangle^{\mathrm{U}(1)})e^{-i\beta}, \\ |P\rangle &\equiv |\mathcal{C}\rangle_{\rm RR+} &&= \tfrac{1}{\sqrt{2}}(|\mathcal{C}_1\rangle^{\mathrm{U}(1)} + |\mathcal{C}_{-1}\rangle^{\mathrm{U}(1)}), \\ |(-)^{F}P\rangle &\equiv |\mathcal{C}\rangle_{\rm RR-} &&= \tfrac{1}{\sqrt{2}}(|\mathcal{C}_1\rangle^{\mathrm{U}(1)} - |\mathcal{C}_{-1}\rangle^{\mathrm{U}(1)}), \end{aligned} \tag{3.15} $$

where the PSS and crosscap Ishibashi states are related by the standard formula

$$ |\mathcal{C}_S\rangle^{\mathrm{U}(1)} \;=\; \sum_{s}\frac{P_{Ss}}{\sqrt{S_{0s}}}\,|\mathcal{C};s\rangle\!\rangle^{\mathrm{U}(1)}, \qquad P_{Ss} \;=\; \frac{\delta^{(2)}_{S,s}}{\sqrt{2}}e^{-\frac{i\pi Ss}{4}}. \tag{3.16} $$

The normalization was chosen to satisfy

$$ \begin{aligned} |\mathcal{C}\rangle_{\rm NSNS\pm} &= e^{i\pi(L_0\pm i\beta\mp\frac{1}{4})}|\mathcal{B}\rangle_{\rm NSNS\pm} &&= e^{\pm i\beta\mp\frac{i\pi}{4}}(\,|\mathcal{C};0\rangle\!\rangle^{\mathrm{U}(1)} \pm i\,|\mathcal{C};2\rangle\!\rangle^{\mathrm{U}(1)}), \\ |\mathcal{C}\rangle_{\rm RR+} &= e^{i\pi(L_0-\frac{1}{8})}|\mathcal{B}\rangle_{\rm RR+} &&= \quad |\mathcal{C};1\rangle\!\rangle^{\mathrm{U}(1)} \;+\; |\mathcal{C};-1\rangle\!\rangle^{\mathrm{U}(1)}, \\ |\mathcal{C}\rangle_{\rm RR-} &= e^{i\pi(L_0-\frac{1}{8})}|\mathcal{B}\rangle_{\rm RR-} &&= -i\,|\mathcal{C};1\rangle\!\rangle^{\mathrm{U}(1)} + i\,|\mathcal{C};-1\rangle\!\rangle^{\mathrm{U}(1)}. \end{aligned} \tag{3.17} $$

Note that these relations ensure that the crosscap condition on fermions are automatically satisfied on the crosscap states.

The arbitrary phase $e^{\pm i\beta}$ in the definition of NSNS crosscaps changes the action of NSNS parities on RR states uniformly by a factor $e^{\pm 2i\beta}$. Such a renormalization is important in constructing orientifolds in superstring theory with real tension. In the following we work with the choice

$$ \beta = \frac{\pi}{4}, $$

so that the NSNS crosscaps have real overlaps with the ground state.

We next construct the permutation crosscap in the orbifold $\mathrm{U}(1)^{\otimes 2}/(\mathbb{Z}_2)^2$ by applying our general prescription given in the previous section. Our starting formula is an adaptation of the formula (2.73) to the orbifold of $\mathrm{U}(1)_2^2$ theory with $q \equiv 0$,

$$|\psi_1^{c_1}\psi_2^{c_2} P_S \pi\rangle \;=\; \sum_s \frac{S_{S\,s+2c_2}}{S_{0s}} R_\circ^{(12)}\, |\mathcal{C}; s+2c_1, s+2c_2\rangle\rangle e^{i\pi\left(2h_{2c_1}+2h_s-h_{s+2c_1}-h_{s+2c_2}\right)}, \tag{3.18}$$

which has the correct overlap (2.68) with the diagonal brane in twisted sectors,

$$|\mathcal{B}_S^{(12)}\rangle^{(\psi_1\psi_2)^c} \;\equiv\; \sum_s \frac{S_{Ss}}{S_{0s}} R_\circ^{(12)}\, |\mathcal{B}; s+2c, s\rangle\rangle. \quad (c=0,1) \tag{3.19}$$

By summing over them weighted by appropriate characters of $(\mathbb{Z}_2)^2$ we find

$$\begin{aligned} |\mathcal{C}^{(12)}\rangle_{\mathrm{NSNS}\pm} &= \frac{1}{2}\sum_{c_1,c_2=0,1} |\psi_1^{c_1}\psi_2^{c_2} P_0\pi\rangle (\pm)^{c_1+c_2} &&= R^{(12)}(|\mathcal{C}\rangle_{\mathrm{NSNS}\pm})^{\otimes 2},\\ |\mathcal{C}^{(12)}\rangle_{\mathrm{RR}\pm} &= \pm\frac{1}{2}\sum_{c_1,c_2=0,1} |\psi_1^{c_1}\psi_2^{c_2} P_1\pi\rangle (\pm)^{c_1}(\mp)^{c_2} &&= i\,R^{(12)}(|\mathcal{C}\rangle_{\mathrm{RR}\pm})^{\otimes 2}. \end{aligned} \tag{3.20}$$

3.3 Parity action on states

The Möbius strip amplitudes of $\mathrm{U}(1)_2/\mathbb{Z}_2$ theory satisfy

$$\begin{aligned} {}_{\mathrm{NSNS},\epsilon}\langle\mathcal{B}|q^{H_c}|\mathcal{C}\rangle_{\mathrm{NSNS}\pm} &= {}_{\mathrm{NSNS}\mp}\langle\mathcal{C}|q^{H_c}|\mathcal{B}\rangle_{\mathrm{NSNS},\epsilon},\\ {}_{\mathrm{RR},\epsilon}\langle\mathcal{B}|q^{H_c}|\mathcal{C}\rangle_{\mathrm{RR}\pm} &= {}_{\mathrm{RR}\mp}\langle\mathcal{C}|q^{H_c}|\mathcal{B}\rangle_{\mathrm{RR},-\epsilon}, \end{aligned} \tag{3.21}$$

from which one can read off the action of parity on some closed string states. The NSNS parities map $|0\otimes 0\rangle, |2\otimes 2\rangle$ to themselves, whereas the RR parities both map $|\pm 1\otimes \mp 1\rangle$ to $\pm i|\mp 1\otimes \pm 1\rangle$.

The action of parity can also be found from the Klein bottle amplitudes. For example, the eigenvalues of $(\pm)^F P$ on RR states are read from

$${}_{\mathrm{RR}\mp}\langle\mathcal{C}|e^{i\pi\tau H_c+i\pi\nu J_0}|\mathcal{C}\rangle_{\mathrm{RR}\pm} \;=\; \pm\, i\chi^{\mathrm{R}-}(\tau,\nu) \;=\; \pm\,\chi^{\mathrm{R}-}(\tau',\nu\tau'). \quad (\tau' = -1/\tau) \tag{3.22}$$

The parameter ν plays the role of a regulator to make amplitudes nonzero. In the tree channel description of Klein bottle, a nonzero ν makes the amplitude finite because $e^{i\pi\nu J_0}|\mathcal{C}\rangle_{\mathrm{RR}+}$ satisfies the rotated crosscap condition,

$$(\tilde{\psi}_n^{\pm} + ie^{i\pi n\pm 2\pi i\nu}\psi_{-n}^{\pm})e^{i\pi\nu J_0}|\mathcal{C}\rangle_{\mathrm{RR}+} \;=\; 0. \tag{3.23}$$

In the loop channel, ν twists the periodicity of the fermion on the circle as

$$\psi^{\pm}(\zeta e^{2\pi i}) = -e^{\pm 2\pi i\nu}\psi^{\pm}(\zeta), \quad \tilde{\psi}^{\pm}(\bar{\zeta}e^{-2\pi i}) = -e^{\mp 2\pi i\nu}\tilde{\psi}^{\pm}(\bar{\zeta}), \tag{3.24}$$

so that their modes $\psi_r^{\pm}, \tilde{\psi}_r^{\pm}$ satisfy $r \in \mathbb{Z}\mp\nu$. This in particular resolves the degeneracy of RR ground states: $|\pm 1\otimes \pm 1\rangle$ have $L_0 = \bar{L}_0 = \frac{1}{8}\pm\frac{\nu}{2}$. The one-loop partition sum in

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such a *spectral flowed* sector should be described by characters with arguments $(\tau', \nu\tau')$. From (3.22) one finds

$$(\pm)^F P|+1 \otimes +1\rangle = \pm|+1 \otimes +1\rangle,$$
$$(\pm)^F P|-1 \otimes -1\rangle = \mp|-1 \otimes -1\rangle.$$

The action of parity thus found is summarized as follows,

$$\begin{array}{rrlrl} & P|0\otimes 0\rangle = & |0\otimes 0\rangle, & P|+1\otimes +1\rangle = & |+1\otimes +1\rangle, \\ \star & P|0\otimes 2\rangle = & i|2\otimes 0\rangle, & P|+1\otimes -1\rangle = & i|-1\otimes +1\rangle, \\ \star & P|2\otimes 0\rangle = & -i|0\otimes 2\rangle, & P|-1\otimes +1\rangle = & -i|+1\otimes -1\rangle, \\ & P|2\otimes 2\rangle = & -|2\otimes 2\rangle, & P|-1\otimes -1\rangle = & -|-1\otimes -1\rangle. \end{array} \tag{3.25}$$

The equations with $\star$ are not obtained from Möbius strip nor Klein bottle, and are chosen by hand so that $P\Psi P = \widetilde{\Psi}$ is satisfied. The analysis of Klein bottles also determines the action of NSNS parities and various fermion number operators on closed string states. The fermion numbers F_R, F_L and F are implicitly defined by the formulae (3.15). The operators $(-)^{F_L, F_R}$ take $(+1)$ on both of $|0 \otimes 0\rangle$ and $|1 \otimes 1\rangle$, and their values on other states follow from the fact that $\Psi, \widetilde{\Psi}$ carry the corresponding fermion number. It also turns out that

$$(-)^{F_L+F_R+F} = 1 \text{ on NSNS states, } (-1) \text{ on RR states.} \tag{3.26}$$

It is a simple exercise to check the action of permutation parity on closed string states; the crosscaps $|\mathcal{C}^{(12)}\rangle_{\mathrm{Y}}$ indeed correspond to the parity $P\pi$, $\pi = (12)$ and its three cousins dressed by fermion number operators. In checking this, note that π gives rise to $(\pm)$ signs when permuting the states of two $\mathrm{U}(1)_2$'s as (2.61).

4. $N = 2$ minimal model

In this section we study the permutation branes and orientifolds in products of $N = 2$ minimal models, which are basic building blocks in Gepner's construction of worldsheet theories of superstring. The $N = 2$ minimal model at level k, which we denote by $M(k)$, is known to be described as simple $N = 2$ supersymmetric LG models of a single chiral field X with superpotential X^{k+2} and a $\mathbb{Z}_{k+2}$ symmetry,

$$\gamma \ : \ X \to e^{\frac{2\pi i}{k+2}} X. \tag{4.1}$$

To construct boundary and crosscap states satisfying suitable conditions on $N = 2$ supercurrents, we start from the *rational* minimal model or the coset model

$$\widehat{\mathrm{SU}(2)}_k \otimes \widehat{\mathrm{U}(1)}_2 \Big/ \widehat{\mathrm{U}(1)}_{k+2}. \tag{4.2}$$

Since all the constituents are purely bosonic, the construction of boundary or crosscap states of the section 2 applies without any problem. On the other hand, the above LG models (which we simply call "$N = 2$ minimal model") are known to be described as different cosets,

$$M(k) \equiv \widehat{\mathrm{SU}(2)}_k \otimes (\text{Dirac fermion}) \Big/ \widehat{\mathrm{U}(1)}_{k+2},$$

so these two cosets are related by the same $\mathbb{Z}_2$-orbifolding as was discussed in the previous section.

The representations of rational minimal model are labelled by three integers (l, m, s) specifying the properties under the affine $\mathrm{SU}(2)_k, \mathrm{U}(1)_{k+2}$ and $\mathrm{U}(1)_2$ respectively. Namely they take values

$$0 \le l \le k, \quad m \simeq m + 2(k+2), \quad s \simeq s + 4.$$

The labels are further restricted by $l + m + s \in 2\mathbb{Z}$, and subject to the *field identification* $(l, m, s) \simeq (k - l, m + k + 2, s + 2)$. Their conformal weight h_{lms} is quadratic in (l, m, s) modulo integer,

$$h_{lms} = \frac{l(l+2) - m^2}{4(k+2)} + \frac{s^2}{8} - \theta(l, m, s), \qquad \theta(l, m, s) \in \mathbb{Z}. \tag{4.3}$$

The functions $\theta(l, m, s)$ and $\sigma_{lms} \equiv e^{i\pi\theta(l,m,s)}$ are nothing but the improvement of conformal weight discussed at section 2.2.1 and equations (2.34), (2.35). See [7] for their precise values. They will be frequently used in constructing crosscap states.

The theory has a $\mathrm{U}(1)$ R-symmetry, and the states in the representation (l, m, s) all have the same R-charge modulo $2\mathbb{Z}$,

$$J_0 \;=\; \frac{m}{k+2} - \frac{s}{2} \quad (\mathrm{mod}\ 2\mathbb{Z}). \tag{4.4}$$

The representations with $l \equiv 0$ are simple currents $g_{m,s}$. They simply shift the m and s quantum numbers when fused with other representations. The simple current $\psi \equiv g_{0,2}$ generates the group $\mathbb{Z}_2$, and the orbifold of rational models by this $\mathbb{Z}_2$ (with $q \equiv 0$) gives the $N = 2$ minimal models. The simple current $\gamma \equiv g_{2,0}$, on the other hand, generates the group $\mathbb{Z}_{k+2}$ which is identified with the phase rotation of the LG field (4.1).

Our aim in this section is to construct quartets of boundary or crosscap states in minimal models and their products corresponding to different spin structures. In terms of the worldsheet $N = 1$ supercurrent they are characterized by

$$\begin{aligned} (\tilde{G}_r \mp i G_{-r})|\mathcal{B}\rangle_{\mathrm{Y}\pm} &= 0, \\ (\tilde{G}_r \mp i e^{i\pi r} G_{-r})|\mathcal{C}\rangle_{\mathrm{Y}\pm} &= 0, \end{aligned} \qquad \begin{cases} r \in \mathbb{Z} + \frac{1}{2} & (\mathrm{Y} = \mathrm{NSNS}) \\ r \in \mathbb{Z} & (\mathrm{Y} = \mathrm{RR}) \end{cases} \tag{4.5}$$

The signs are flipped when the states are multiplied by the operators $(-)^{F_R}$ or $(-)^{F_L}$.

In $N = 2$ SCFTs, one can instead use the operators $e^{i\pi J_0}$ or $e^{i\pi \tilde{J}_0}$ to flip the sign, where $J_0, \tilde{J}_0$ are the right, left-moving R-charges. Moreover, the NSNS and RR states are related by spectral flow. Let us denote by U a combination of left-right spectral flows acting on the generators of two $N = 2$ SCAs as

$$\begin{aligned} U J_n U^{-1} = J_n + \tfrac{\hat{c}}{2}\delta_{n,0},\ U G_n^{\pm} U^{-1} = G^{\pm}_{n\pm 1/2},\ U L_n U^{-1} = L_n + \tfrac{1}{2} J_n + \tfrac{\hat{c}}{8}\delta_{n,0}, \\ U \tilde{J}_n U^{-1} = \tilde{J}_n - \tfrac{\hat{c}}{2}\delta_{n,0},\ U \tilde{G}_n^{\pm} U^{-1} = \tilde{G}^{\pm}_{n\mp 1/2},\ U \tilde{L}_n U^{-1} = \tilde{L}_n - \tfrac{1}{2} \tilde{J}_n + \tfrac{\hat{c}}{8}\delta_{n,0}. \end{aligned} \tag{4.6}$$

U maps a closed string state in $\mathcal{V}_{l,m,s} \otimes \mathcal{V}_{l,\tilde{m},\tilde{s}}$ to a state in $\mathcal{V}_{l,m+1,s+1} \otimes \mathcal{V}_{l,\tilde{m}-1,\tilde{s}-1}$. It is easy to see that U or $U e^{-i\pi J_0/2}$ map the NSNS solutions of boundary or crosscap conditions to

RR solutions. We assign a phase φ to each of the quartet states as follows,

$$\begin{aligned} U|\mathcal{B}\rangle_{\rm NSNS+} &= |\mathcal{B}\rangle_{\rm RR+} e^{-i\pi\varphi(\mathcal{B})}, \\ U e^{-i\pi J_0/2}|\mathcal{C}\rangle_{\rm NSNS+} &= |\mathcal{C}\rangle_{\rm RR+} e^{-i\pi\varphi(\mathcal{C})}. \end{aligned} \tag{4.7}$$

In type II superstring theory, the phase φ of D-branes and orientifolds characterizes the unbroken spacetime $\mathcal{N}=1$ supersymmetry.

4.1 Boundary and crosscap states

Boundary or crosscap states $|\mathcal{B}_{L,M,S}\rangle$, $|\mathcal{C}_{0,M,S}\rangle$ in rational minimal models are constructed from Ishibashi states $|\mathcal{B};l,m,s\rangle\rangle$, $|\mathcal{C};l,m,s\rangle\rangle$ in the standard way. The $\mathbb{Z}_2$-orbifolding reorganizes them into solutions of suitable boundary or crosscap conditions on supercurrent. For boundary states, we define the Ishibashi states solving the boundary conditions on supercurrents as follows,

$$\begin{aligned} |\mathcal{B};l,m\rangle\rangle_{\rm NSNS\pm} &= |\mathcal{B};l,m,0\rangle\rangle \pm |\mathcal{B};l,m,2\rangle\rangle, \\ |\mathcal{B};l,m\rangle\rangle_{\rm RR+} &= |\mathcal{B};l,m,1\rangle\rangle + |\mathcal{B};l,m,-1\rangle\rangle, \\ |\mathcal{B};l,m\rangle\rangle_{\rm RR-} &= -i\,|\mathcal{B};l,m,1\rangle\rangle + i\,|\mathcal{B};l,m,-1\rangle\rangle, \end{aligned} \tag{4.8}$$

whereas for crosscaps the appropriate combinations of Ishibashi states are

$$\begin{aligned} |\mathcal{C};l,m\rangle\rangle_{\rm NSNS\pm} &= e^{\pi i(L_0-h_{lm0})}\sigma_{lm0}\,|\mathcal{B};l,m\rangle\rangle_{\rm NSNS\pm}, \\ |\mathcal{C};l,m\rangle\rangle_{\rm RR\pm} &= e^{\pi i(L_0-h_{lm1})}\sigma_{lm1}\,|\mathcal{B};l,m\rangle\rangle_{\rm RR\pm}, \end{aligned} \tag{4.9}$$

where $\sigma_{lms}\equiv e^{i\pi\theta(l,m,s)}$ is given at (4.3), or more explicitly

$$\begin{aligned} |\mathcal{C};l,m\rangle\rangle_{\rm NSNS\pm} &= \sigma_{lm0}\,|\mathcal{C};l,m,0\rangle\rangle \pm i\sigma_{lm2}\,|\mathcal{C};l,m,2\rangle\rangle, \\ |\mathcal{C};l,m\rangle\rangle_{\rm RR+} &= \sigma_{lm1}\,|\mathcal{C};l,m,1\rangle\rangle + \sigma_{lm-1}\,|\mathcal{C};l,m,-1\rangle\rangle, \\ |\mathcal{C};l,m\rangle\rangle_{\rm RR-} &= -i\sigma_{lm1}\,|\mathcal{C};l,m,1\rangle\rangle + i\sigma_{lm-1}\,|\mathcal{C};l,m,-1\rangle\rangle. \end{aligned} \tag{4.10}$$

The D-branes and orientifolds in $N=2$ minimal model are given by a sum over $\mathbb{Z}_2$-orbit of rational boundaries or crosscaps [7, 12]. In terms of the above Ishibashi states they read

$$\begin{aligned} |\mathcal{B}_{L,M}\rangle_{\rm Y} &= \frac{1}{2}\sum_{l,m}\frac{S_{\rm LM}^{\;lm}}{\sqrt{S_{00}^{lm}}}\,|\mathcal{B};l,m\rangle\rangle_{\rm Y}, \\ |\mathcal{C}_M\rangle_{\rm Y} &= \frac{\beta_{M,\rm Y}}{2}\sum_{(l,m)}\frac{P_{0M}^{\;lm}}{\sqrt{S_{00}^{\;lm}}}\,|\mathcal{C};l,m\rangle\rangle_{\rm Y} \;=\; \frac{1}{2}\sum_{(l,m)}\frac{P_{k,M+k+2}^{\;lm}}{\sqrt{S_{00}^{\;lm}}}\,|\mathcal{C};l,m\rangle\rangle_{\rm Y}, \\ \beta_{M,\rm NSNS\pm} &= \mp i(-)^{\frac{M}{2}},\quad \beta_{M,\rm RR\pm}=(-)^{\frac{M\pm1}{2}}. \end{aligned} \tag{4.11}$$

Here (l,m) runs over integers $0\le l\le k,\ m\sim m+2k+4$. The S and P matrices are twice the product of those of $\mathrm{SU}(2)_k$ and $(\mathrm{U}(1)_{k+2})^*$ theories,

$$\begin{aligned} S_{ll'} &= \sqrt{\tfrac{2}{k+2}}\sin\pi\tfrac{(l+1)(l'+1)}{k+2}, & S_{mm'} &= \tfrac{1}{\sqrt{2k+4}}e^{\frac{i\pi mm'}{k+2}}, \\ P_{ll'} &= \sqrt{\tfrac{4}{k+2}}\delta^{(2)}_{k+l+l'}\sin\pi\tfrac{(l+1)(l'+1)}{2k+4}, & P_{mm'} &= \tfrac{\delta^{(2)}_{k+m+m'}}{\sqrt{k+2}}e^{\frac{i\pi mm'}{2k+4}}. \end{aligned} \tag{4.12}$$

The coefficients $\beta_{M,\mathrm{Y}}$ are introduced mainly for later convenience, but it also has some physical significances. For one thing, they make the states $|\mathcal{C}_M\rangle_{\mathrm{NSNS}\pm}$ periodic and $|\mathcal{C}_M\rangle_{\mathrm{RR}\pm}$ anti-periodic under $M \to M+2k+4$, so that the shift of M by $2k+4$ is regarded as the orientation flip. It also preserves the action of simple current γ on crosscap states, so we have

$$\gamma|\mathcal{B}_{L,M}\rangle_{\mathrm{Y}} \;=\; |\mathcal{B}_{L,M+2}\rangle_{\mathrm{Y}}, \qquad \gamma|\mathcal{C}_M\rangle_{\mathrm{Y}} \;=\; |\mathcal{C}_{M+4}\rangle_{\mathrm{Y}}. \tag{4.13}$$

The spectral flow U for $N=2$ minimal models is identified with the fusion with the simple current $g_{1,1}$. The boundary and crosscap states of minimal model are then shown to form the following quartets,

$$\begin{aligned}
(1+e^{i\pi J_0})(1+e^{i\pi\varphi(\mathcal{B}_{L,M})}U)|\mathcal{B}_{L,M}\rangle_{\mathrm{NSNS}+} &= |\mathcal{B}_{L,M}\rangle_{\mathrm{NSNS}+} + |\mathcal{B}_{L,M+1}\rangle_{\mathrm{NSNS}-}\\
&\quad + |\mathcal{B}_{L,M}\rangle_{\mathrm{RR}+} + |\mathcal{B}_{L,M+1}\rangle_{\mathrm{RR}-},\\
(1+e^{i\pi J_0})(1+e^{i\pi\varphi(\mathcal{C}_M)}Ue^{-\frac{i\pi J_0}{2}})|\mathcal{C}_M\rangle_{\mathrm{NSNS}+} &= |\mathcal{C}_M\rangle_{\mathrm{NSNS}+} + |\mathcal{C}_{M+2}\rangle_{\mathrm{NSNS}-}\\
&\quad + |\mathcal{C}_{M-1}\rangle_{\mathrm{RR}+} + |\mathcal{C}_{M+1}\rangle_{\mathrm{RR}-},
\end{aligned} \tag{4.14}$$

with $\varphi(\mathcal{B}_{L,M}) = \frac{M}{k+2}$, $\varphi(\mathcal{C}_M) = \frac{M-1}{2k+4} + \frac{1}{2}$.

4.1.1 Boundary states in $g_{k+2,2}$ twisted sector

When k is even, the boundary states with $L = k/2$ are fixed by $g_{k+2,2} \equiv \eta$. We define the boundary states sitting in η-twisted sector [12],

$$\begin{aligned}
|\widetilde{\mathcal{B}}_{k/2,M,S}\rangle^{\eta} &= \frac{1}{2}\sum_{(ms)} \frac{\widetilde{S}_{k/2\,MS}^{\;k/2\,ms}}{\sqrt{S_{000}^{k/2\,ms}}}\,|\mathcal{B};\tfrac{k}{2},m,s\rangle\!\rangle^{\eta},\\
\widetilde{S}_{k/2\,MS}^{\;k/2\,ms} &= 2S_{Mm}S_{Ss}e^{-\frac{i\pi}{2}(M-S+m-s)}.
\end{aligned} \tag{4.15}$$

The boundary conditions on supercurrent are solved by the following linear combinations of Ishibashi states,

$$\begin{aligned}
|\tfrac{k}{2},m\rangle\!\rangle^{\eta}_{\mathrm{NSNS}+} &= |\tfrac{k}{2},m,0\rangle\!\rangle^{\eta} - |\tfrac{k}{2},m,2\rangle\!\rangle^{\eta},\\
|\tfrac{k}{2},m\rangle\!\rangle^{\eta}_{\mathrm{NSNS}-} &= i\,|\tfrac{k}{2},m,0\rangle\!\rangle^{\eta} + i\,|\tfrac{k}{2},m,2\rangle\!\rangle^{\eta},\\
|\tfrac{k}{2},m\rangle\!\rangle^{\eta}_{\mathrm{RR}+} &= i\,|\tfrac{k}{2},m,1\rangle\!\rangle^{\eta} - i\,|\tfrac{k}{2},m,-1\rangle\!\rangle^{\eta},\\
|\tfrac{k}{2},m\rangle\!\rangle^{\eta}_{\mathrm{RR}-} &= i\,|\tfrac{k}{2},m,1\rangle\!\rangle^{\eta} + i\,|\tfrac{k}{2},m,-1\rangle\!\rangle^{\eta}.
\end{aligned} \tag{4.16}$$

Note the sign difference in taking linear combinations as compared to (4.8) due to the difference in Grassmann parity. The corresponding quartet of boundary states is given by

$$|\mathcal{B}_{k/2,M}\rangle^{\eta}_{\mathrm{Y}} \;=\; \frac{1}{2}\sum_{m} \frac{\widetilde{S}_{k/2\,M}^{\;k/2\,m}}{\sqrt{S_{00}^{k/2\,m}}}\,|\mathcal{B};\tfrac{k}{2},m\rangle\!\rangle^{\eta}_{\mathrm{Y}}, \tag{4.17}$$

where

$$\widetilde{S}_{k/2\,M}^{\;k/2\,m} \;=\; 2S_{Mm}e^{-\frac{i\pi}{2}(M+m)}. \tag{4.18}$$

After the orbifold by $\mathbb{Z}_2$ is taken, there is no distinction in labelling the twisted sector by $g_{k+2,2}$ or $g_{k+2,0}$. We therefore use the symbol η for the simple current $g_{k+2,0}$ in what follows.

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4.1.2 Tension and charge

The tension and RR charges of D-brane and orientifolds are given by the overlaps of the boundary or crosscap states with the NSNS and RR vacua. We denote the NSNS chiral primary states and RR ground states as,

$$|l_{\rm NS}\rangle \;=\; |(l,l,0)\otimes(l,-l,0)\rangle, \quad |l_{\rm R}\rangle \;=\; |(l,l+1,1)\otimes(l,-l-1,-1)\rangle. \tag{4.19}$$

The overlaps of these states with boundary or crosscap states read [12]

$$\begin{aligned}
\langle l_{\rm R}|\mathcal{B}_{L,M}\rangle_{\rm RR+} &= e^{\frac{i\pi M}{k+2}}\cdot\langle l_{\rm NS}|\mathcal{B}_{L,M}\rangle_{\rm NSNS+} \;=\; \frac{e^{\frac{i\pi M(l+1)}{k+2}}\sin\frac{\pi(L+1)(l+1)}{k+2}}{\sqrt{\frac{k+2}{2}\sin\frac{\pi(l+1)}{k+2}}},\\
\langle l_{\rm R}|\mathcal{C}_{2m-1}\rangle_{\rm RR+} &= e^{\frac{i\pi(2m-l-1)}{2k+4}+\frac{i\pi}{2}}\cdot\langle l_{\rm NS}|\mathcal{C}_{2m}\rangle_{\rm NSNS+}\\
&= \begin{cases}\langle l_{\rm R}|\mathcal{B}_{\frac{k}{2},\frac{2m+k+1-(-)^m}{2}}\rangle_{\rm RR+} & (k\ \text{ even}),\\ \langle l_{\rm R}|\mathcal{B}_{\frac{k+(-)^m}{2},\frac{2m+k+1}{2}}\rangle_{\rm RR+} & (k\ \text{ odd}).\end{cases}
\end{aligned} \tag{4.20}$$

Tensions are therefore given by

$$\begin{aligned}
\langle 0_{\rm NS}|\mathcal{B}_{L,M}\rangle_{\rm NSNS\pm} &= T_0\sin\frac{\pi(L+1)}{k+2},\\
\langle 0_{\rm NS}|\mathcal{C}_{2m}\rangle_{\rm NSNS\pm} &= \begin{cases}T_0 e^{-\frac{i\pi(-)^m}{2k+4}}, & (k\ \text{even})\\ T_0\cos\frac{\pi}{2k+4}, & (k\ \text{odd})\end{cases}
\end{aligned} \tag{4.21}$$

where $T_0=\left(\frac{k+2}{2}\sin\frac{\pi}{k+2}\right)^{-\frac{1}{2}}$.

4.1.3 Parity action on closed string states

Klein bottle amplitude gives a lot of information on the action of parity on closed string states in minimal model or its orbifolds. We take an arbitrary orbifold group $\Gamma\subset\mathbb{Z}_{k+2}$ and consider orientifolds in the orbifold,

$$|\mathcal{C}_{M,r}\rangle_{\rm Y} \;=\; \frac{1}{\sqrt{|\Gamma|}}\sum_{\gamma^\nu\in\Gamma}|\mathcal{C}_{M+2\nu}\rangle_{\rm Y}\exp\left(-\frac{2\pi i\nu r}{k+2}\right). \tag{4.22}$$

The parameter r labels the dressing by quantum symmetry that multiplies phases to different twisted sectors, and $2r$ has to be even for NSNS states and odd for RR states because of the (anti-)periodicity of the crosscap states in M. The parity $P_{M,r}$ corresponding to $|\mathcal{C}_{M,r}\rangle_{\rm RR+}$, as well as its cousins, are in general all non-involutive and square to some quantum symmetry. The action of $P_{M,r}$ on closed string states has to be of the form

$$P_{M,r}|(l,m,s)\otimes(l,\tilde{m},\tilde{s})\rangle \;=\; |(l,\tilde{m},\tilde{s})\otimes(l,m,s)\rangle\exp\left(\frac{i\pi(\tilde{m}+m)2r+i\pi(\tilde{m}-m)M}{2k+4}\right)p_{s,\tilde{s}}. \tag{4.23}$$

The Klein bottle amplitudes show that this is indeed the case, and moreover $p_{s,\tilde{s}}$ are given by

$$\begin{aligned}
p_{0,0}=p_{1,1} &= 1, & p_{0,2}=p_{1,-1} &= -i,\\
p_{2,2}=p_{-1,-1} &= -1, & p_{2,0}=p_{-1,1} &= i.
\end{aligned} \tag{4.24}$$

The other three crosscaps with $Y = \text{NSNS}\pm$, RR$-$ are corresponding to the parity $P_{M,r}$ combined with the fermion numbers $(-)^{F_L,F_R,F}$ satisfying (3.26). Comparisons of various Klein bottle amplitudes determine the values of these fermion numbers; the states with $s = \tilde{s} = 0$ or 1 have $(-)^{F_R} = (-)^{F_L} = 1$, and their values on other states follow from the obvious rules.

Using these results one can derive the action of parity on boundary states. For those in the untwisted sector we have

$$\begin{aligned} (-)^{F_L} P_{\bar{M},r} |\mathcal{B}_{L,M}\rangle_{\text{NSNS}\pm} &= \quad |\mathcal{B}_{L,\bar{M}-M}\rangle_{\text{NSNS}\pm}, \\ P_{\bar{M},r} |\mathcal{B}_{L,M}\rangle_{\text{RR}\pm} &= -|\mathcal{B}_{L,\bar{M}-M}\rangle_{\text{RR}\mp}. \end{aligned} \tag{4.25}$$

This agrees with the transformation law obtained from Möbius strip amplitudes (1.7). The boundary states in η-twisted sector are transformed as follows:

$$\begin{aligned} (-)^{F_L} P_{\bar{M},r} |\mathcal{B}_{k/2,M}\rangle^{\eta}_{\text{NSNS}\pm} &= \mp i e^{i\pi r} |\mathcal{B}_{k/2,\bar{M}-M}\rangle^{\eta}_{\text{NSNS}\pm}, \\ P_{\bar{M},r} |\mathcal{B}_{k/2,M}\rangle^{\eta}_{\text{RR}\pm} &= \quad e^{i\pi r} |\mathcal{B}_{k/2,\bar{M}-M}\rangle^{\eta}_{\text{RR}\mp}. \end{aligned} \tag{4.26}$$

4.2 Permutation branes

It is straightforward to construct permutation branes in the tensor products of N minimal models. We start by the permutation boundary states in the product of N rational minimal model and take $(\mathbb{Z}_2)^N$-orbifold. We give the expression for those corresponding to the cyclic permutation of length N, i.e. $\pi = (12\cdots N)$.

$$|\mathcal{B}^{(12\cdots N)}_{L,M}\rangle_{\text{Y}} \equiv \frac{\alpha_{\text{Y}}}{2} \sum_{l,m} \frac{S_{\text{LM}}^{\ lm}}{(S_{00}^{\ lm})^{N/2}} R^{(12\cdots N)}\, |\mathcal{B}; (l,m)^{\otimes N}\rangle\!\rangle_{\text{Y}},$$

$$\alpha_{\text{NSNS}\pm} = 1, \qquad \alpha_{\text{RR}\pm} = i^{N-1}. \tag{4.27}$$

Recalling the case of $\mathrm{U}(1)_2$ where we have to sum over rational boundary states of odd S-labels when N is even, we find that the labels (L, M) obey

$$\begin{aligned} (N \ \ \text{odd}) \implies & \begin{cases} L + M = (\text{even}) & \text{for NSNS+, RR+ states,} \\ L + M = (\text{odd}) & \text{for NSNS}-,\text{RR}-\text{ states,} \end{cases} \\ (N \ \ \text{even}) \implies & \ L + M = (\text{odd}) \quad \text{for all states,} \end{aligned}$$

The simple current $\otimes_a \gamma_a^{\nu_a}$ shifts their M-label by $2\sum_a \nu_a$. In particular, the simple currents with $\sum_a \nu_a = 0 \bmod (k+2)$ fix the boundary states. The states $|\mathcal{B}^{(1\cdots N)}_{L,M}\rangle_{\text{Y}+}, |\mathcal{B}^{(1\cdots N)}_{L,M+N}\rangle_{\text{Y}-}$ form a quartet with the phase $\varphi = \frac{M}{k+2} + \frac{1-N}{2}$.

The RR charges of permutation branes are given by the overlaps with the states $|l_{\text{R}}^{\otimes N}\rangle$,

$$\langle l_{\text{R}}^{\otimes N} | \mathcal{B}^{(12\cdots N)}_{L,M}\rangle_{\text{RR}+} = \frac{\sin \frac{\pi(L+1)(l+1)}{k+2}\, e^{\frac{i\pi M(l+1)}{k+2} + \frac{i\pi(1-N)}{2}}}{(\frac{k+2}{2})^{1-\frac{N}{2}} (\sin \frac{\pi(l+1)}{k+2})^{N/2}}. \tag{4.28}$$

The tension is given by

$$\langle 0_{\text{NS}}^{\otimes N} | \mathcal{B}^{(12\cdots N)}_{L,M}\rangle_{\text{NSNS}+} = (\tfrac{k+2}{2})^{\frac{N}{2}-1} (\sin \tfrac{\pi}{k+2})^{-\frac{N}{2}} \sin \tfrac{\pi(L+1)}{k+2}. \tag{4.29}$$

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4.3 Permutation orientifolds

Here we construct the permutation crosscaps for tensor products of two identical minimal models through the $(\mathbb{Z}_2)^2$-orbifold procedure. Denoting by $\psi_{1,2}$ the simple currents $g_{0,2}$ in the two copies of minimal model, we sum over the following crosscaps (with $\pi = (1\,2)$)

$$
\begin{aligned}
|\psi_1^{c_1}\psi_2^{c_2}P_{M,S}\pi\rangle = \frac{1}{2}\sum_{l,m,s}\frac{S_{0MS}^{\;lm\,s+2c_2}}{S_{000}^{\;lms}}R_\circ^\pi\,|\mathcal{C};(l,m,s+2c_1),(l,m,s+2c_2)\rangle\!\rangle \\
\times \exp i\pi\left\{2h_{0,0,2c_1}+2h_{l,m,s}-h_{l,m,s+2c_1}-h_{l,m,s+2c_2}\right\}
\end{aligned}
\tag{4.30}
$$

with appropriate weight to obtain

$$
\begin{aligned}
|\mathcal{C}^\pi_M\rangle_{\text{NSNS}\pm} &= \frac{1}{2}\sum_{c_i=0,1}|\psi_1^{c_1}\psi_2^{c_2}P_{M,0}\pi\rangle(\pm)^{c_1+c_2} = \frac{1}{2}\sum_{l,m}\frac{S_{0M}^{\;lm}}{S_{00}^{\;lm}}R^\pi\,|\mathcal{C};(l,m)^{\otimes 2}\rangle\!\rangle_{\text{NSNS}\pm} \\
|\mathcal{C}^\pi_M\rangle_{\text{RR}\pm} &= \pm\frac{1}{2}\sum_{c_i=0,1}|\psi_1^{c_1}\psi_2^{c_2}P_{M,1}\pi\rangle(\pm)^{c_1}(\mp)^{c_2} = \frac{i}{2}\sum_{l,m}\frac{S_{0M}^{\;lm}}{S_{00}^{\;lm}}R^\pi\,|\mathcal{C};(l,m)^{\otimes 2}\rangle\!\rangle_{\text{RR}\pm}.
\end{aligned}
\tag{4.31}
$$

Note that M is even for NSNS states and odd for RR states. One can furthermore consider the parities $\gamma_1^{\nu_1}\gamma_2^{\nu_2}P_{M,S}$ which are non-involutive for general $\nu_{1,2}$. The corresponding crosscap states are obtained by applying the formula (2.73),

$$
|\mathcal{C}^\pi_{M+2\nu_1,M+2\nu_2}\rangle_{\text{Y}} = \frac{\alpha_{\text{Y}}}{2}\sum_{l,m}\frac{S_{0,M+\nu_1+\nu_2}^{\;l,m+\nu_1+\nu_2}}{S_{00}^{\;lm}}R^\pi\,|\mathcal{C};(l,m+2\nu_1),(l,m+2\nu_2)\rangle\!\rangle_{\text{Y}}.
\tag{4.32}
$$

Here $\alpha_{\text{NSNS}\pm}=1,\ \alpha_{\text{RR}\pm}=i$.

We thus constructed the crosscap states $|\mathcal{C}^{(12)}_{M_1,M_2}\rangle$ for different spin structures; the labels $M_{1,2}$ are both even and periodic under $(2k+4)$-shift for NSNS crosscaps, while they are both odd and anti-periodic for RR crosscaps. The simple current $\gamma_1^{\nu_1}\gamma_2^{\nu_2}$ shifts both of the labels M_1, M_2 by $2\nu_1+2\nu_2$. They are organized into quartets satisfying

$$
\begin{aligned}
(1+e^{i\pi J_0})(1+e^{i\pi\varphi}Ue^{-i\pi J_0/2})|\mathcal{C}^{(12)}_{M_1,M_2}\rangle_{\text{NSNS}+} &= |\mathcal{C}^{(12)}_{M_1,M_2}\rangle_{\text{NSNS}+} + |\mathcal{C}^{(12)}_{M_1+2,M_2+2}\rangle_{\text{NSNS}-} \\
&\quad + |\mathcal{C}^{(12)}_{M_1-1,M_2-1}\rangle_{\text{RR}+} + |\mathcal{C}^{(12)}_{M_1+1,M_2+1}\rangle_{\text{RR}-},
\end{aligned}
\tag{4.33}
$$

with $\varphi = \frac{M_1+M_2-2}{2k+4} - \frac{1}{2}$. The RR charges and tension are given by

$$
\begin{aligned}
\langle l_{\text{R}}^{\otimes 2}|\mathcal{C}^{(12)}_{M-1,M-1}\rangle_{\text{RR}+} &= \langle l_{\text{R}}^{\otimes 2}|\mathcal{B}^{(12)}_{0,M-1}\rangle_{\text{RR}+} = e^{\frac{i\pi(M-1)(l+1)}{k+2}-\frac{i\pi}{2}}, \\
\langle 0_{\text{NS}}^{\otimes 2}|\mathcal{C}^{(12)}_{M,M}\rangle_{\text{NSNS}+} &= \langle 0_{\text{NS}}^{\otimes 2}|\mathcal{B}^{(12)}_{0,M}\rangle_{\text{NSNS}+} = 1.
\end{aligned}
\tag{4.34}
$$

The permutation crosscaps with $M_1 \neq M_2$ are tensionless, but they have nonzero overlaps with RR vacua sitting in twisted sectors. Let us define

$$
|l_{\text{R}}^{\text{tw}}\rangle \equiv |(l,l+1,1)\otimes(l,l+1,1)\rangle.
\tag{4.35}
$$

Then one finds

$$
\langle (k-l)_{\text{R}}^{\text{tw}}\otimes l_{\text{R}}^{\text{tw}}|\mathcal{C}^{(12)}_{M,M+2l+2}\rangle_{\text{RR}+} = -i.
\tag{4.36}
$$

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5. Gepner models

We apply the results of the preceding sections to the construction of permutation D-branes and orientifolds in Gepner models, which are type II superstring theories defined from orbifolds of products of $N=2$ minimal models and affine $\mathrm{U}(1)_2$ models [1].

Gepner's original construction of the models starts with a product of r rational minimal models and d copies of affine $\mathrm{U}(1)_2$ models, and then takes its orbifold by a group of simple currents. A subgroup $\tilde{\Gamma}_{\mathrm{GSO}} \simeq (\mathbb{Z}_2)^{r+d-1}$ of this orbifold is formed by even monomials of the simple currents $\psi_1, \cdots, \psi_{r+d}$ discussed in previous sections that shift the s quantum numbers by two. As we have reviewed in detail in the previous sections, this is equivalent to taking the product of r $N=2$ minimal models and d Dirac fermions and then summing over spin structures. For constructing D-branes and orientifolds, this just amounts to taking the product of boundary or crosscap states with the sector index Y aligned. In this way one can focus on the r minimal models describing the internal manifold separately from the part describing the noncompact spacetime.

It only remains to explain the "rest" of the Gepner's orbifold group. Gepner models describe the CFT on certain Calabi-Yau D-folds at special points in the moduli space in terms of orbifolds of products of r minimal models. The central charges of constituent minimal models therefore add up to $3D$,

$$\sum_{a=1}^{r} \frac{3k_a}{k_a+2} = 3D. \tag{5.1}$$

We also assume without losing generality that

$$r - D \ = \ \text{even}, \tag{5.2}$$

since we can add minimal models with $k=0$. The product of minimal models is orbifolded by $\Gamma = \mathbb{Z}_H$ ($H \equiv \mathrm{lcm}(k_a+2)$) generated by $\gamma_{(A)} \equiv \prod_{a=1}^{r} \gamma_a$ to ensure the integrality of R-charge. The orbifold is taken according to the standard simple current prescription of section 2.2 with

$$q(\gamma_a, \gamma_b) = \frac{\delta_{ab}}{k_a+2}.$$

Gepner model $\otimes_{a=1}^{r} M(k_a)/\Gamma$ is mirror to a different orbifold $\otimes_{a=1}^{r} M(k_a)/\Gamma_{\mathrm{mir}}$, where

$$\Gamma_{\mathrm{mir}} \ \equiv \ \left\{ \prod_a \gamma_a^{m_a};\ \sum_a \frac{m_a}{k_a+2} \in \mathbb{Z} \right\}. \tag{5.3}$$

In particular, B-branes (B-type orientifolds) in the original Gepner model are mirror of the A-branes (A-type orientifolds) in the mirror Gepner model and vise versa.

Examples. We denote various Gepner models by the set of integers (k_a+2). Two main examples of Gepner models we discuss in this paper are the model (55555) corresponding to a quintic hypersurface in $\mathbb{CP}^4$, and (88444) corresponding to an octic hypersurface in weighted projective space $\mathbb{WCP}^4_{1,1,2,2,2}$. These models have been extensively studied because of small $h_{1,1}$ of the corresponding Calabi-Yau spaces.

We describe the D-branes or orientifolds in superstring theory by suitable linear combinations of quartet states of the worldsheet CFT,

$$\begin{aligned} 2|\mathcal{B}\rangle &= |\mathcal{B}\rangle_{\mathrm{NSNS}+} - |\mathcal{B}\rangle_{\mathrm{NSNS}-} + |\mathcal{B}\rangle_{\mathrm{RR}+} - |\mathcal{B}\rangle_{\mathrm{RR}-}, \\ 2|\mathcal{C}\rangle &= -i|\mathcal{C}\rangle_{\mathrm{NSNS}+} + i|\mathcal{C}\rangle_{\mathrm{NSNS}-} + |\mathcal{C}\rangle_{\mathrm{RR}+} - |\mathcal{C}\rangle_{\mathrm{RR}-}. \end{aligned} \tag{5.4}$$

Here the quartet states are given by the products of the states from the internal and spacetime CFTs,

$$|\mathcal{B}\rangle_{\mathrm{Y}} = |\mathcal{B}\rangle^{\mathrm{int}}_{\mathrm{Y}} \otimes |\mathcal{B}\rangle^{\mathrm{st}}_{\mathrm{Y}}, \qquad |\mathcal{C}\rangle_{\mathrm{Y}} = |\mathcal{C}\rangle^{\mathrm{int}}_{\mathrm{Y}} \otimes |\mathcal{C}\rangle^{\mathrm{st}}_{\mathrm{Y}}. \tag{5.5}$$

The spacetime parts $|\mathcal{B}\rangle^{\mathrm{st}}_{\mathrm{Y}}, |\mathcal{C}\rangle^{\mathrm{st}}_{\mathrm{Y}}$ contain the fields for $\mathbb{R}^{2d+2}$ as well as ghosts [40, 2], and are normalized to produce consistent one-loop amplitudes. In particular, they satisfy

$$\begin{aligned} (-)^{F_L}|\mathcal{B}\rangle_{\mathrm{Y}+} &= |\mathcal{B}\rangle_{\mathrm{Y}-}, \qquad (-)^{F_L}|\mathcal{C}\rangle_{\mathrm{Y}+} = |\mathcal{C}\rangle_{\mathrm{Y}-}, \\ |\mathcal{C}\rangle_{\mathrm{Y}} &= 2^{d+1}\exp i\pi(L_0 - h^{\mathrm{st}}_{\mathrm{Y}})|\mathcal{B}\rangle_{\mathrm{Y}}. \qquad (h^{\mathrm{st}}_{\mathrm{NS}} = -\tfrac{1}{2},\ \ h^{\mathrm{st}}_{\mathrm{R}} = \tfrac{d-4}{8}) \end{aligned} \tag{5.6}$$

The normalization of the internal parts are fixed from the integrality of various one-loop amplitudes. Alternatively, it is determined by requiring that the NSNS states $|\mathcal{B}\rangle^{\mathrm{int}}_{\mathrm{NSNS}\pm}$, $|\mathcal{C}\rangle^{\mathrm{int}}_{\mathrm{NSNS}\pm}$ have real overlaps with the ground state of the internal CFT. Such overlaps appear as coefficients of the dilaton tadpole and are regarded as the tensions of D-branes or orientifolds. The overlaps with various RR ground states measure the RR charges. The sign flip of the RR part of $|\mathcal{B}\rangle$ or $|\mathcal{C}\rangle$ therefore gives anti-D-branes or anti-orientifolds.

One can compute cylinder, Möbius strip and Klein bottle amplitudes between various D-branes and orientifolds as overlaps of the states $|\mathcal{B}\rangle$ and $|\mathcal{C}\rangle$. In doing this, remember that the simple dagger of a ket state for a D-brane or orientifold gives a bra state for anti-D-brane or anti-orientifold.

Tadpole cancellation. Consistent configurations of D-branes $\mathcal{B}_i$ and orientifold $\mathcal{C}$ in superstring theory must be free of RR tadpoles [40, 41], namely, the tadpole state

$$|\mathcal{T}\rangle = |\mathcal{C}\rangle + \sum_i |\mathcal{B}_i\rangle, \tag{5.7}$$

must not have any overlaps with massless RR scalar states. The non-vanishing tadpoles of massless NSNS scalars do not lead to inconsistency [42]. However, the absence of RR tadpoles automatically guarantees that NSNS tadpoles also vanish if the configuration of D-branes and orientifolds preserves a spacetime supersymmetry. The spacetime $\mathcal{N} = 2$ supersymmetry is related to worldsheet spectral flows in the left and right-moving sectors, and the phase φ (4.7) determines the $N = 1$ supersymmetry unbroken by the branes or orientifolds. So $|\mathcal{T}\rangle$ preserves spacetime supersymmetry if all the boundary and crosscap states in $|\mathcal{T}\rangle$ are labelled by one and the same phase φ.

The absence of NSNS tadpoles for supersymmetric tadpole-free configurations is shown by noticing that the massless NSNS and RR states are related to the chiral primary and

RR ground states in the internal CFT, and are therefore paired up by spectral flow. For each of such pairs we can show

$$\frac{\langle l_{\rm R}|\mathcal{B}_i\rangle}{\langle l_{\rm NS}|\mathcal{B}_i\rangle} = \exp i\pi\varphi,$$
$$\frac{\langle l_{\rm R}|\mathcal{C}\rangle}{\langle l_{\rm NS}|\mathcal{C}\rangle} = i\exp i\pi\left[\varphi - \tfrac{1}{2}J_0^{\rm int}(l_{\rm NS}) - \{L_0^{\rm st}(l_{\rm NS}) - (h_0^{\rm st})_{\rm NS}\}\right] \;=\; \exp i\pi\varphi. \tag{5.8}$$

Here we used $\frac{1}{2}J_0^{\rm int} + L_0^{\rm st} = L_0^{\rm int+st} = 0$ for the state $l_{\rm NS}$ of our interest, and chose a suitable normalization for $l_{\rm R}$. It immediately follows from this that

$${}_{\rm RR}\langle l|\mathcal{T}\rangle \;=\; e^{i\pi\varphi}\,{}_{\rm NSNS}\langle l|\mathcal{T}\rangle, \tag{5.9}$$

for tadpole states $|\mathcal{T}\rangle$ preserving spacetime supersymmetry characterized by the phase φ.

Remark. in our convention (2.1) of boundary or crosscap conditions, the $N = 2$ supercurrents $G^{\pm}$ are glued to $\tilde{G}^{\pm}$ along the A-branes or A-type orientifolds though they are usually called B-type conditions.

5.1 Permutation D-branes in Gepner models

We turn to construct and classify permutation branes in Gepner models. They were constructed in [16] and studied in [21 − 25]. Here we give a construction of them based on the simple current orbifold prescription, paying particular attention to those labelled by $L = k/2$ which require a special care. We study the A-type branes first, and then study the B-type branes using the mirror description.

5.1.1 A-branes

A-branes in Gepner models are labelled by a permutation π and (L_c, M_c) with $c = 1, \cdots, [\pi]$, where $[\pi]$ denotes the number of cycles in π and $\|\pi_c\|$ the length of the cycle π_c. The branes with trivial stabilizer group are simply given by summing over $\mathbb{Z}_H$-images,

$$|\mathcal{B}^{A,\pi}_{\mathbf{L},\mathbf{M}}\rangle \;=\; \frac{1}{\sqrt{H}}\sum_{\nu\in\mathbb{Z}_H}|\mathcal{B}^{\pi}_{\mathbf{L},\gamma^{\nu}_{(A)}(\mathbf{M})}\rangle \;\equiv\; \frac{1}{\sqrt{H}}\sum_{\nu\in\mathbb{Z}_H}\otimes_{c=1}^{[\pi]}|\mathcal{B}^{\pi_c}_{L_c,M_c+2\nu\|\pi_c\|}\rangle. \tag{5.10}$$

Here and in the following the index for spin structure will be suppressed whenever possible. The label $(\mathbf{L},\mathbf{M})$ contains some redundancy because different values of $\mathbf{M}$ related by $\mathbb{Z}_H$-shifts label the same D-brane, and the following change of the label $(\mathbf{L},\mathbf{M})$

$$\mathcal{F}_c \;:\; (\cdots L_c \cdots \;;\; \cdots M_c \cdots) \;\to\; (\cdots k_c - L_c \cdots \;;\; \cdots M_c + k_c + 2 \cdots), \tag{5.11}$$

maps $|\mathcal{B}^{A,\pi}_{\mathbf{L},\mathbf{M}}\rangle$ to its anti-brane.

Some A-branes with special choices of π or $\mathbf{L}$ have nontrivial stabilizer groups. The boundary state (5.10) are invariant under $\gamma^{H'}_{(A)}$ $(H' < H)$ if

$$\frac{H'\|\pi_c\|}{k_c+2} \in \mathbb{Z} \;\;\text{for all}\;\; c. \tag{5.12}$$

Such branes should be defined as sums over twists as well as over images. Moreover, if H' is even, the boundary states are invariant also under $\gamma_{(A)}^{H'/2}$ if

$$L_c = \frac{k_c}{2} \quad \text{for all } c \text{ such that } \; w'_c \equiv \frac{H'\|\pi_c\|}{k_c+2} \; \text{ is odd.} \tag{5.13}$$

These D-branes are generalization of short-orbit branes discussed in detail in [12]. To see how the enhancement of the stabilizer occurs, note first that $\gamma_{(A)}^{H'/2}$ shifts M_c by k_c+2 when w'_c is odd, and acts trivially on other M_c's. Therefore, with the help of the maps $\mathcal{F}_c$, $\gamma_{(A)}^{H'/2}$ maps the brane satisfying (5.13) to itself or its antibrane depending on how many of w'_c are odd. Since there are always an even number of odd w'_c under the condition (5.2) the branes satisfying (5.13) are always mapped to themselves by $\gamma_{(A)}^{H'/2}$.

To write down the branes with nontrivial stabilizers, we first introduce the boundary states in twisted sectors of the product of N minimal models following (2.41) and (2.42),

$$\begin{aligned} |\mathcal{B}_{L,M}^{(12\cdots N)}\rangle_{\mathrm{Y}}^{(\mu)} &= \frac{\alpha_{\mathrm{Y}}}{2}\sum_{l,m}\frac{S_{\mathrm{LM}}^{\;lm}}{(S_{00}^{lm})^{N/2}}\,|\mathcal{B}^{(12\cdots N)};l,m\rangle\!\rangle_{\mathrm{Y}}^{(\mu)}, \\ |\mathcal{B}^{(12\cdots N)};l,m\rangle\!\rangle_{\mathrm{Y}}^{(\mu)} &= R^{(12\cdots N)}\,|\mathcal{B};(l,m+2\mu)\otimes(l,m+4\mu)\otimes\cdots\otimes(l,m)\rangle\!\rangle_{\mathrm{Y}}. \end{aligned} \tag{5.14}$$

Here α_{Y} is defined in (4.27). The label of twisted sectors μ satisfies $\mu N \in (k+2)\mathbb{Z}$. When the level k is even and $\mu N \in (k+2)(\mathbb{Z}+\frac{1}{2})$, we define

$$|\widetilde{\mathcal{B}}_{k/2,M}^{(12\cdots N)}\rangle_{\mathrm{Y}}^{(\mu)} = \frac{\alpha_{\mathrm{Y}}}{2}\sum_{l,m}\frac{\widetilde{S}_{k/2M}^{\;k/2\,m}}{(S_{00}^{k/2,m})^{N/2}}\,|\widetilde{\mathcal{B}}^{(12\cdots N)};\tfrac{k}{2},m\rangle\!\rangle_{\mathrm{Y}}^{(\mu)}, \tag{5.15}$$

$$|\widetilde{\mathcal{B}}^{(12\cdots N)};\tfrac{k}{2},m\rangle\!\rangle_{\mathrm{Y}}^{(\mu)} = R^{(12\cdots N)}\,|\mathcal{B};(\tfrac{k}{2},m+2\mu)\otimes(\tfrac{k}{2},m+4\mu)\otimes\cdots\otimes(\tfrac{k}{2},m+k+2)^{\eta}\rangle\!\rangle_{\mathrm{Y}}.$$

The tilde will be omitted in what follows unless we need to distinguish the states (5.15) from (5.14). The boundary states invariant under $\gamma_{(A)}^{h}$ $(hH'=H)$ take the form

$$|\mathcal{B}_{\mathbf{L},\mathbf{M}}^{A,\pi,\rho}\rangle \equiv \frac{1}{\sqrt{H}}\sum_{\nu\in\mathbb{Z}_h,\;\mu\in\mathbb{Z}_{H'}}\otimes_{c=1}^{[\pi]}|\mathcal{B}_{L_c,M_c+2\nu\|\pi_c\|}^{\pi_c}\rangle^{(\mu h)}\exp\left(\frac{2\pi i\rho\mu h}{H}\right). \tag{5.16}$$

Here $\rho\in\mathbb{Z}_{H'}$ specifies a character of the stabilizer group.

Example 1: (55555)

The π-permuted boundary states have nontrivial stabilizer when $\prod_{a\in\pi_c}\gamma_a = 1$ for all cycles of π, namely, all the cycles of π have the lengths divisible by 5. Therefore, $\pi = (12345)$ is up to conjugation the only case with nontrivial stabilizer $\mathcal{H} = \mathbb{Z}_5$. The untwisted stabilizer is $\mathcal{H}$ itself, so the boundary states are sums over $\mathbb{Z}_5$-twists.

Example 2: (88444)

There are D-branes with various stabilizer groups. Generic non-permuted A-branes do not have stabilizers, while those with $L_1 = L_2 = 3$ are invariant under $\gamma_{(A)}^4$. Generic π-permuted A-branes are invariant under $\gamma_{(A)}^4$ when π permutes $a = 1, 2$. Some of such D-branes are invariant under $\gamma_{(A)}^2$ if their L-labels satisfy (5.13). For all these cases, the untwisted stabilizer agrees with the stabilizer itself.

π	$\mathcal{H}$ (generator)	$\mathcal{U}$ (generator)
(1)(2)(3)(4)(5)	1	1
(12)(3)(4)(5)	$\mathbb{Z}_5$ $(\gamma_1\gamma_2^4)$	$\mathbb{Z}_5$ $(\gamma_1\gamma_2^4)$
(12)(34)(5)	$(\mathbb{Z}_5)^2$ $(\gamma_1\gamma_2^4, \gamma_3\gamma_4^4)$	$(\mathbb{Z}_5)^2$ $(\gamma_1\gamma_2^4, \gamma_3\gamma_4^4)$
(123)(4)(5)	$(\mathbb{Z}_5)^2$ $(\gamma_1\gamma_2^4, \gamma_2\gamma_3^4)$	1
(123)(45)	$(\mathbb{Z}_5)^3$ $(\gamma_1\gamma_2^4, \gamma_2\gamma_3^4, \gamma_4\gamma_5^4)$	$\mathbb{Z}_5$ $(\gamma_4\gamma_5^4)$
(1234)(5)	$(\mathbb{Z}_5)^3$ $(\gamma_1\gamma_2^4, \gamma_2\gamma_3^4, \gamma_3\gamma_4^4)$	$\mathbb{Z}_5$ $(\gamma_1\gamma_2^4\gamma_3\gamma_4^4)$
(12345)	$(\mathbb{Z}_5)^4$ $(\gamma_1\gamma_2^4, \gamma_2\gamma_3^4, \gamma_3\gamma_4^4, \gamma_4\gamma_5^4)$	1

Table 1: B-branes of the model (55555) and their stabilizer $\mathcal{H}$, untwisted stabilizer $\mathcal{U}$.

5.1.2 B-branes

We would like to study B-branes in Gepner model using the mirror description with the orbifold group $\Gamma_{\rm mir}$ of (5.3). The label of D-branes consists of a permutation π and quantum numbers $(L_c, M_c)\,(c = 1, \cdots, [\pi])$, as well as a character of its untwisted stabilizer group. Since the label $\mathbf{M}$ has a large redundancy due to the shifts by elements of $\Gamma_{\rm mir}$, we sometimes use

$$M \;\equiv\; \sum_{c=1}^{[\pi]} m_c w_c \qquad \left(w_c \equiv \frac{H}{k_c+2}\right), \tag{5.17}$$

There is also a map $\mathcal{F}_c$ (5.11) that sends a brane to its antibrane.

In mirror Gepner model there are indeed branes with different (untwisted) stabilizer groups. We first focus on generic permutation branes with none of L_c coinciding with $k_c/2$. They start to have nontrivial stabilizer group as soon as π becomes nontrivial. If π contains a cycle $\pi_c = (1\,2 \cdots N)$, then all the π-permuted branes are fixed by $(\mathbb{Z}_{k_c+2})^{N-1}$,

$$\mathcal{H} \;\supset\; (\mathbb{Z}_{k_c+2})^{N-1} \equiv \{\gamma_1^{\nu_1}\gamma_2^{\nu_2}\cdots\gamma_N^{\nu_N} \;|\; \sum_i \nu_i \in (k_c+2)\mathbb{Z}\}.$$

So the generic π-permuted branes have stabilizer $\mathcal{H} = \otimes_{c=1}^{[\pi]}(\mathbb{Z}_{k_c+2})^{\|\pi_c\|-1}$.

By analyzing its action on twisted sectors using (2.44), one finds that none of the the stabilizer $(\mathbb{Z}_{k_c+2})^{N-1}$ contributes to the untwisted subgroup $\mathcal{U}$ for odd N, while a $\mathbb{Z}_{k_c+2}$ subgroup generated by $\gamma_1\gamma_2^{-1}\cdots\gamma_N^{-1}$ contributes to $\mathcal{U}$ for even N. As an example we list the permutation B-branes of the model (55555) with their (untwisted) stabilizers in the table below.

The permutation branes with nontrivial untwisted stabilizers are made from permutation boundary states $|\mathcal{B}_{L,M}^{(12\cdots N),\rho}\rangle$ in the orbifold $M(k)^N/\Gamma_{\rm mir}$, where N is even and

$$\Gamma_{\rm mir} \;=\; (\mathbb{Z}_{k+2})^{N-1} \;=\; \{\gamma_1^{\nu_1}\cdots\gamma_N^{\nu_N}\,|\sum \nu_a = 0 \mod (k+2)\}. \tag{5.18}$$

The label ρ specifies a character of the untwisted stabilizer $\mathbb{Z}_{k+2}$ generated by $\gamma_1\gamma_2^{-1}\cdots\gamma_N^{-1}$.

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We find it convenient to define the boundary states in terms of Ishibashi states as

$$|\mathcal{B}^{(12\cdots N),\rho}_{L,M}\rangle_{\rm Y} = \frac{1}{\sqrt{k+2}}\sum_{\nu}\exp\left(\tfrac{2\pi i\rho\nu}{k+2}\right)\sum_{l,m}\frac{\alpha_{\rm Y}}{2}\frac{S_{\rm LM}^{\ lm}}{(S_{00}^{lm})^{N/2}} \times R_{(12\cdots N)}\,|\mathcal{B};(l,m+\nu)\otimes(l,m-\nu)\otimes\cdots\otimes(l,m-\nu)\rangle\rangle_{\rm Y}, \quad (5.19)$$

where $\alpha_{\rm Y}$ is defined in (4.27). It is easy to check the following,

$$|\mathcal{B}^{(12\cdots N),\rho}_{L,M}\rangle = |\mathcal{B}^{(2\cdots N1),-\rho}_{L,M}\rangle, \qquad \gamma_a|\mathcal{B}^{(12\cdots N),\rho}_{L,M}\rangle = |\mathcal{B}^{(12\cdots N),\rho}_{L,M+2}\rangle. \quad (5.20)$$

However, due to the non-standard definition of the Ishibashi states in twisted sectors, ρ has to be integer or half-odd integer depending on whether M is even or odd. One also finds

$$\begin{aligned} |\mathcal{B}^{(12\cdots N),\,\rho}_{L,M}\rangle_{\rm NSNS\pm} &= |\mathcal{B}^{(12\cdots N),\,\rho+\frac{k+2}{2}}_{k-L,M+k+2}\rangle_{\rm NSNS\pm}, \\ |\mathcal{B}^{(12\cdots N),\,\rho}_{L,M}\rangle_{\rm RR\pm} &= -|\mathcal{B}^{(12\cdots N),\,\rho+\frac{k+2}{2}}_{k-L,M+k+2}\rangle_{\rm RR\pm}. \end{aligned} \quad (5.21)$$

As an example, the permutation B-branes in (55555) model for $\pi=(12)(34)$ is given by

$$|\mathcal{B}^{B,(12)(34),\,\rho,\,\rho''}_{\mathbf{L},\mathbf{M}}\rangle = \frac{1}{5}\sum_{\nu+\nu'+\nu''\in 5\mathbb{Z}}|\mathcal{B}^{(12),\,\rho}_{L,M+2\nu}\rangle\otimes|\mathcal{B}^{(34),\,\rho'}_{L',M'+2\nu''}\rangle\otimes|\mathcal{B}^{(5)}_{L'',M''+2\nu''}\rangle. \quad (5.22)$$

Next we discuss the enhancement of stabilizer group when some of k_c are even and $L_c = k_c/2$. A permutation brane labelled by π and $\{L_1,\cdots,L_{[\pi]}\}$ is invariant under the following simple currents

$$\begin{aligned} &(i)\ \ \gamma_a\gamma_b^{-1}\ \ (a,b \text{ are in the same cycle}) \\ &(ii)\ \eta_a\eta_b\ \ \ \ (a,b \text{ are in cycles labelled by } L=k/2). \end{aligned} \quad (5.23)$$

So the stabilizer group for a permutation brane gets enhanced by $(\mathbb{Z}_2)^{n-1}$ if n (≥ 2) cycles of π are labelled by $L_c = k_c/2$. The **L**-label of B-branes is called *special* (or *generic*) if two or more (resp. at most one) of L_c coincide with $k_c/2$.

It is a little intricate to find out the untwisted stabilizer for these short-orbit branes. For the D-branes with $\pi = $ id and $L_a = k_a/2$ for $a=1,\cdots,n$, the boundary states in twisted sectors should be expressed as products of $|\mathcal{B}_{k_a/2,M_a}\rangle_{\rm Y}$ and $|\mathcal{B}_{k_a/2,M_a}\rangle^{\eta_a}_{\rm Y}$. However, the action of $\eta=\gamma^{\frac{k+2}{2}}$ on boundary states in the untwisted and η-twisted sectors differ by a sign,

$$\begin{aligned} \eta|\mathcal{B}_{k/2,M}\rangle_{\rm NSNS\pm} &= +|\mathcal{B}_{k/2,M}\rangle_{\rm NSNS\pm}, & \eta|\mathcal{B}_{k/2,M}\rangle^{\eta}_{\rm NSNS\pm} &= -|\mathcal{B}_{k/2,M}\rangle^{\eta}_{\rm NSNS\pm}, \\ \eta|\mathcal{B}_{k/2,M}\rangle_{\rm RR\pm} &= -|\mathcal{B}_{k/2,M}\rangle_{\rm RR\pm}, & \eta|\mathcal{B}_{k/2,M}\rangle^{\eta}_{\rm RR\pm} &= +|\mathcal{B}_{k/2,M}\rangle^{\eta}_{\rm RR\pm}. \end{aligned} \quad (5.24)$$

So the only states invariant under all the elements (ii) of the stabilizer group (5.23) are those in the untwisted sector and $(\eta_1\cdots\eta_n)$-twisted sector. The latter exists only when n is even. The untwisted stabilizer for non-permuted branes is given by

$$\mathcal{U} = \begin{cases} 1 & (n \text{ odd}) \\ \mathbb{Z}_2 = \{1,\prod_{a=1}^n \eta_a\} & (n \text{ even}) \end{cases}. \quad (5.25)$$

π	$\sharp(L_c = k_c/2)$	$\mathbf{L}$	$\mathcal{H}$	$\mathcal{U}$
(12)(345)	2	$(3,1)$	$\mathbb{Z}_8 \times (\mathbb{Z}_4)^2 \times \mathbb{Z}_2$	$\mathbb{Z}_4\ (\gamma_1^2\gamma_2^6)$
	≤ 1	any	$\mathbb{Z}_8 \times (\mathbb{Z}_4)^2$	$\mathbb{Z}_8\ (\gamma_1\gamma_2^7)$
(1)(2)(345)	3	$(3,3,1)$	$(\mathbb{Z}_4)^2 \times (\mathbb{Z}_2)^2$	1
	2	$(3,*,1)$	$(\mathbb{Z}_4)^2 \times \mathbb{Z}_2$	$\mathbb{Z}_2\ (\eta_1\eta_3\eta_4\eta_5)$
	2	$(3,3,*)$	$(\mathbb{Z}_4)^2 \times \mathbb{Z}_2$	$\mathbb{Z}_2\ (\eta_1\eta_2)$
	≤ 1	any	$(\mathbb{Z}_4)^2$	1
(12)(34)(5)	3	$(3,1,1)$	$\mathbb{Z}_8 \times \mathbb{Z}_4 \times (\mathbb{Z}_2)^2$	$\mathbb{Z}_4 \times \mathbb{Z}_2\ (\gamma_1^2\gamma_2^6, \gamma_3^2\gamma_4^2)$
	2	$(3,1,*)$	$\mathbb{Z}_8 \times \mathbb{Z}_4 \times \mathbb{Z}_2$	$\mathbb{Z}_8 \times \mathbb{Z}_2\ (\gamma_1\gamma_2^7\gamma_3\gamma_4^3, \gamma_3^2\gamma_4^2)$
	2	$(3,*,1)$	$\mathbb{Z}_8 \times \mathbb{Z}_4 \times \mathbb{Z}_2$	$\mathbb{Z}_4 \times \mathbb{Z}_4\ (\gamma_1^2\gamma_2^6, \gamma_3\gamma_4^3)$
	2	$(*,1,1)$	$\mathbb{Z}_8 \times \mathbb{Z}_4 \times \mathbb{Z}_2$	$\mathbb{Z}_8 \times \mathbb{Z}_2\ (\gamma_1\gamma_2^7, \gamma_3^2\gamma_4^2)$
	≤ 1	any	$\mathbb{Z}_8 \times \mathbb{Z}_4$	$\mathbb{Z}_8 \times \mathbb{Z}_4\ (\gamma_1\gamma_2^7, \gamma_3\gamma_4^3)$

Table 2: Some permutation B-branes in the model (88444).

Generalizing this to permutation branes, one finds the following result. For each even-length cycle $\pi_c = (a_1, a_2, \ldots, a_{2l})$ of π, denote by γ_{π_c} the following simple current

$$\gamma_{\pi_c} \;=\; \gamma_{a_1}\gamma_{a_2}^{-1}\gamma_{a_3}\cdots\gamma_{a_{2l}}^{-1}. \tag{5.26}$$

Then the untwisted stabilizer for permutation branes with $L_c = k_c/2$ for more than one cycles is generated by the following:

1. γ_{π_c}, where π_c is an even-length cycle labelled by $L_c \neq k_c/2$,
2. $(\gamma_{\pi_c})^2$, where π_c is an even-length cycle labelled by $L_c = k_c/2$,
3. The element
$$\check{\gamma} \;\equiv\; \left(\prod_a \eta_a\right)\left(\prod_{L_c=k_c/2}\gamma_{\pi_c}\right), \tag{5.27}$$
where the first product is over all a's belonging to odd-length cycles labelled by $L_c = k_c/2$, and the second is over all even-length cycles π_c labelled by $L_c = k_c/2$. This is an element of $\Gamma_{\rm mir}$ only when there are even number of odd-length cycles labelled by $L_c = k_c/2$.

Interestingly, when L_c's coincide with $k_c/2$ the untwisted stabilizer group gets reduced due to 1 $\to$ 2 of the list, and then enhances by 3 of the list.

As an example, we list some of the permutation B-branes, their stabilizers and untwisted stabilizers in the model (88444).

Let us pick up some examples from the list and illustrate the construction of boundary states. We first take the case $\pi = (1)(2)(345)$, which is a rather straightforward generalization of non-permuted branes because all the cycles have odd length. The π-permuted B-branes split into two when $\mathbf{L} = (3,*,1)$. To describe the boundary states in $\eta_1\eta_3\eta_4\eta_5$-twisted sector, we use the states $|\mathcal{B}_{k/2,M}\rangle^\eta$ defined at section 4.1.1 and their generalization to arbitrary odd-length cycles,

$$|\widetilde{\mathcal{B}}^{(12\cdots N)}_{k/2,M}\rangle \;\equiv\; |\mathcal{B}^{(12\cdots N)}_{k/2,M}\rangle^{\eta_1\cdots\eta_N}_{\rm Y} \;=\; \frac{\alpha_{\rm Y}}{2}\sum_{l,m}\frac{\widetilde{S}_{k/2,M}^{\;lm}}{(S^{lm}_{00})^{N/2}}R_{(12\cdots N)}\,|\mathcal{B};\otimes_{a=1}^{N}(l,m)^{\eta}\rangle\!\rangle_{\rm Y}. \qquad (5.28)$$

Next we study the case $\pi = (12)(34)(5)$. The untwisted stabilizer group for π-permuted B-branes gets smaller as the number of L_c's coinciding with $k_c/2$ increases. We wish to understood this in terms of the boundary states defined at (5.19). For generic $\mathbf{L}$ the branes are defined as

$$|\mathcal{B}^{B,\pi,(\rho,\rho')}_{\mathbf{L},\mathbf{M}}\rangle \;=\; \frac{1}{4}\sum_{\nu+2\nu'+2\nu''\in 8\mathbb{Z}}|\mathcal{B}^{(12),\rho}_{L,M+2\nu}\rangle\otimes|\mathcal{B}^{(34),\rho'}_{L',M'+2\nu'}\rangle\otimes|\mathcal{B}^{(5)}_{L'',M''+2\nu''}\rangle, \qquad (5.29)$$

with the integers $\rho,\ \rho'$ specifying a character of the untwisted stabilizer $\mathbb{Z}_8\times\mathbb{Z}_4$. When some L_c's coincide with $k_c/2$, then the sum over orbifold images is partially translated into the sum over shifts of (ρ,ρ') due to (5.21). When $\mathbf{L}=(3,1,1)$ one can write

$$|\mathcal{B}^{B,\pi,(\rho,\rho')}_{\mathbf{L},\mathbf{M}}\rangle \;=\; \frac{1}{4}\sum_{\nu+2\nu'+2\nu''\in 8\mathbb{Z}}|\mathcal{B}^{(12),\,\rho,+}_{3,M+2\nu}\rangle\otimes|\mathcal{B}^{(34),\,\rho',+}_{1,M'+2\nu'}\rangle\otimes|\mathcal{B}^{(5)}_{1,M''+2\nu''}\rangle, \qquad (5.30)$$

where we define, for any cyclic permutation π of even length,

$$|\mathcal{B}^{\pi,\rho,\pm}_{k/2,M}\rangle \;\equiv\; \frac{1}{2}\left(|\mathcal{B}^{\pi,\,\rho}_{k/2,M}\rangle \pm |\mathcal{B}^{\pi,\,\rho+(k+2)/2}_{k/2,M}\rangle\right). \qquad (5.31)$$

The periodicity of ρ,ρ' thus becomes halved when $\mathbf{L}=(3,1,1)$, in accordance with the untwisted stabilizer becoming smaller for these branes. Note also that the states (5.31) are transformed by η_a's in a similar way as $|\mathcal{B}_{k/2,M}\rangle$ and $|\mathcal{B}_{k/2,M}\rangle^{\eta}$ of (5.24).

When $\mathbf{L}=(3,1,0)$ one can write

$$\begin{aligned}|\mathcal{B}^{B,\pi,(\rho,\rho',\varepsilon)}_{\mathbf{L},\mathbf{M}}\rangle = &\frac{1}{4}\sum_{\nu+2\nu'+2\nu''\in 8\mathbb{Z}}|\mathcal{B}^{(12),\,\rho,+}_{3,M+2\nu}\rangle\otimes|\mathcal{B}^{(34),\,\rho',+}_{1,M'+2\nu'}\rangle\otimes|\mathcal{B}^{(5)}_{0,M''+2\nu''}\rangle\\ &+\frac{\varepsilon}{4}\sum_{\nu+2\nu'+2\nu''\in 8\mathbb{Z}}|\mathcal{B}^{(12),\,\rho,-}_{3,M+2\nu}\rangle\otimes|\mathcal{B}^{(34),\,\rho',-}_{1,M'+2\nu'}\rangle\otimes|\mathcal{B}^{(5)}_{0,M''+2\nu''}\rangle. \qquad (5.32)\end{aligned}$$

The untwisted stabilizer is twice as big as the previous case due to the generator $\check{\gamma}$ (5.27).

5.2 Permutation orientifolds in Gepner model

We next construct and classify the permutation orientifolds in Gepner models. The basic building blocks are the quartets of crosscap states $|\mathcal{C}_M\rangle_{\rm Y}$ (4.11) or $|\mathcal{C}^{(12)}_{M_1,M_2}\rangle_{\rm Y}$ (4.32) defined before. The A-type permutation orientifolds are constructed as sums of their products with characters of $\Gamma_O \equiv \Gamma/(\Gamma\Gamma^{\pi})$, where Γ is the Gepner's orbifold group and

$$\Gamma\Gamma^{\pi} \;\equiv\; \{g\pi g\pi | g\in\Gamma\}.$$

B-type orientifolds are constructed in a similar way using the mirror description. Below we give a general construction, and illustrate it in a few examples.

5.2.1 A-type orientifolds

The orbifold group is $\Gamma = \mathbb{Z}_H$ and one easily finds that

$$\Gamma\Gamma^\pi = \Gamma^2 \equiv \{g^2 | g \in \Gamma\},$$

for any models and any π. Therefore, $\Gamma_O \equiv \Gamma/\Gamma^2$ is a $\mathbb{Z}_2$ for even H and otherwise trivial. We denote by $|\mathcal{C}^\pi_{\mathbf{M}}\rangle_{\rm Y}$ the products of crosscaps $|\mathcal{C}_M\rangle_{\rm Y}$ and $|\mathcal{C}^{(12)}_{M_1,M_2}\rangle_{\rm Y}$ in minimal models. The A-type crosscaps in Gepner models are given by their sums,

$$|\mathcal{C}^{A,\pi,\epsilon}_{\mathbf{M}}\rangle_{\rm Y} \;=\; \frac{c_{\rm Y}}{\sqrt{H}}\sum_\nu \epsilon^\nu |\mathcal{C}^\pi_{\gamma^\nu(\mathbf{M})}\rangle_{\rm Y} \;\equiv\; \frac{c_{\rm Y}}{\sqrt{H}}\sum_\nu \epsilon^\nu |\mathcal{C}^\pi_{\mathbf{M}+\mathbf{2}\nu}\rangle_{\rm Y}. \tag{5.33}$$

with suitable normalization constants $c_{\rm NS}, c_{\rm R}$. The following crosscap states form a quartet,

$$|\mathcal{C}^{A,\pi,\epsilon}_{\mathbf{M}}\rangle_{\rm NSNS+},\quad |\mathcal{C}^{A,\pi,\epsilon}_{\mathbf{M+2}}\rangle_{\rm NSNS-},\quad |\mathcal{C}^{A,\pi,\tilde\epsilon}_{\mathbf{M-1}}\rangle_{\rm RR+},\quad |\mathcal{C}^{A,\pi,\tilde\epsilon}_{\mathbf{M+1}}\rangle_{\rm RR-},$$
$$\tilde\epsilon \equiv \epsilon\cdot\exp\left(-\textstyle\sum_{a=1}^r \frac{i\pi}{k_a+2}\right). \tag{5.34}$$

with the supersymmetry phase

$$\exp i\pi\varphi \;=\; \frac{c_{\rm R}}{c_{\rm NS}}\exp i\pi\left(\sum_{a=1}^5 \frac{(M_a-1)}{2k_a+4} + \frac{r+|\pi|}{2}\right). \tag{5.35}$$

Here $|\pi|$ counts the number of cycles of length two in π. The four possible choices of $c_{\rm NS,R}$ correspond to orientifold planes $O^\pm$ of positive or negative tension, and their anti-planes. The label ϵ can take ± 1 for even H, while only $\epsilon = +1$ is allowed for odd H.

The constant $c_{\rm R}$ takes values ± 1, whereas the correct values of $c_{\rm NS}$ depends on the label ϵ. When H is odd, the tension T of the orientifold is given by $c_{\rm NS}$ up to a positive proportionality constant so we should set $c_{\rm NS} = \pm 1$. When H is even, T becomes proportional to

$$T \;\sim\; c_{\rm NS}\left(e^{-i\pi\Theta_{\mathbf{M}}} + \epsilon e^{+i\pi\Theta_{\mathbf{M}}}\right), \qquad \Theta_{\mathbf{M}} \equiv \sum_{c\ (k_c = {\rm even}, \|\pi_c\|=1)} \frac{(-)^{\frac{M_c}{2}}}{2k_c+4}.$$

So the correct choices of $c_{\rm NS}$ are

$$\begin{array}{l} H = {\rm odd},\ (\epsilon = +) \implies c_{\rm NS} = \pm 1,\ T \sim \pm 1, \\ H = {\rm even},\ \epsilon = + \implies c_{\rm NS} = \pm 1,\ T \sim \pm\cos\pi\Theta_{\mathbf{M}}, \\ H = {\rm even},\ \epsilon = - \implies c_{\rm NS} = \pm i,\ T \sim \pm\sin\pi\Theta_{\mathbf{M}}. \end{array} \tag{5.36}$$

Orientifolds labelled by different $\mathbf{M}$ are related to one another by the global symmetry generated by simple currents,

$$(\otimes_{a=1}^r \gamma_a^{\nu_a})|\mathcal{C}^{A,\pi,\epsilon}_{\mathbf{M}}\rangle \;=\; |\mathcal{C}^{A,\pi,\epsilon}_{\mathbf{M}'}\rangle, \qquad M'_a \equiv M_a + 2\nu_a + 2\nu_{\pi(a)}. \tag{5.37}$$

If H is odd, then any $\mathbf{M}$ can be mapped to $\mathbf{M} = 0$ by the global symmetry. For even H there are several choices for $\mathbf{M}$ that lead to physically inequivalent orientifolds. An interesting fact is that, for even H, the involutiveness of parity does not require $M_a = M_{\pi(a)}$. The condition that the square of parity is an element of Γ implies the existence of a mod-H integer ν satisfying

$$M_a - M_{\pi(a)} = 2\nu \mod 2(k_a+2). \tag{5.38}$$

Since the left hand side is antisymmetric under $a \to \pi(a)$ and the right hand side is symmetric, the only allowed ν are 0 or $H/2$.

Example 1: (55555) There are three involutive permutations of five elements up to conjugation, namely $\pi = \mathrm{id}$, (12) or (12)(34). We denote various products of crosscap states as

$$\begin{aligned} |\mathcal{C}_{\mathbf{M}}^{(1)(2)(3)(4)(5)}\rangle_{\mathrm{Y}} &= |\mathcal{C}_{M_1}^{(1)} \otimes \mathcal{C}_{M_2}^{(2)} \otimes \mathcal{C}_{M_3}^{(3)} \otimes \mathcal{C}_{M_4}^{(4)} \otimes \mathcal{C}_{M_5}^{(5)}\rangle_{\mathrm{Y}}, \\ |\mathcal{C}_{\mathbf{M}}^{(12)(3)(4)(5)}\rangle_{\mathrm{Y}} &= |\mathcal{C}_{M_1,M_2}^{(12)} \otimes \mathcal{C}_{M_3}^{(3)} \otimes \mathcal{C}_{M_4}^{(4)} \otimes \mathcal{C}_{M_5}^{(5)}\rangle_{\mathrm{Y}}, \\ |\mathcal{C}_{\mathbf{M}}^{(12)(34)(5)}\rangle_{\mathrm{Y}} &= |\mathcal{C}_{M_1,M_2}^{(12)} \otimes \mathcal{C}_{M_3,M_4}^{(34)} \otimes \mathcal{C}_{M_5}^{(5)}\rangle_{\mathrm{Y}}. \end{aligned} \tag{5.39}$$

The A-type crosscaps in Gepner model are given by their sums. For the parities to be involutive, we have to set $M_1 = M_2$ in the second line and $M_1 = M_2, M_3 = M_4$ in the third line.

Since all the levels are odd, the crosscaps with different values of $\mathbf{M}$ are all related to the one with $\mathbf{M} = \mathbf{0}$ by global symmetry (simple currents). Moreover, Γ_O is trivial because H is odd. Therefore, there are just three physically inequivalent A-type orientifolds in this model $|\mathcal{C}_{\mathbf{0}}^{A,\pi}\rangle$ labelled by three different permutations. The same argument apply to all other Gepner models with odd H.

Example 2: (88444) In this model there are four inequivalent permutations up to conjugation, namely $\pi = \mathrm{id}, (12), (34)$ or $(12)(34)$. The orientifolds are also labelled by a character of $\Gamma_O = \mathbb{Z}_2$. In order for the orientifold $|\mathcal{C}_{\mathbf{M}}^{A,\pi,\epsilon}\rangle$ to correspond to an involutive parity, the M labels have to satisfy $M_3 = M_4$ if π contains a cycle (34), and $M_1 = M_2$ or $M_1 = M_2 + 8$ if π contains (12). Different values of $\mathbf{M}$ are related by the actions of global symmetry, but this time there remain several choices for $\mathbf{M}$ leading to inequivalent orientifolds. The physically inequivalent choices of labels $(\pi, \mathbf{M})$ are as listed below:

$$\begin{array}{lll} \pi = \mathrm{id}, & \mathbf{M} = & (00000),\ (02000),\ (22000), \\ & & (00002),\ (02002),\ (22002), \\ \pi = (12), & \mathbf{M} = & (00000),\ (00002),\ (08000),\ (08002), \\ \pi = (34), & \mathbf{M} = & (00000),\ (02000),\ (22000), \\ \pi = (12)(34), & \mathbf{M} = & (00000),\ (08000). \end{array} \tag{5.40}$$

The crosscaps containing $|\mathcal{C}_{M_1,M_1+8}^{(12)}\rangle$ are supported only on closed string Hilbert space in the $\gamma_1^4\gamma_2^4$-twisted sector, so they are in particular tensionless. On the other hand, they do have nonzero overlaps with RR ground states in $\gamma_1^4\gamma_2^4$-twisted sector.

5.2.2 B-type permutation orientifolds

We study the B-type permutation orientifolds in Gepner models as A-types in the mirror. The orientifolds are given by summing the crosscap states $|\mathcal{C}_{\mathbf{M}}^{\pi}\rangle_{\mathrm{Y}}$ of the product theory over an orbit of Γ_{mir} weighted by various characters of $\Gamma_O \equiv \Gamma_{\mathrm{mir}}/(\Gamma_{\mathrm{mir}}\Gamma_{\mathrm{mir}}^{\pi})$,

$$|\mathcal{C}_{\mathbf{M}}^{B,\pi,\rho}\rangle_{\mathrm{Y}} = \frac{c_{\mathrm{Y}}}{\sqrt{|\Gamma_{\mathrm{mir}}|}} \sum_{\gamma=\otimes_a\gamma_a^{\nu_a}\in\Gamma_{\mathrm{mir}}} |\mathcal{C}_{\gamma(\mathbf{M})}^{\pi}\rangle_{\mathrm{Y}}\,\rho(\vec{\nu}), \tag{5.41}$$

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where $\gamma(\mathbf{M}) \equiv (M_1+2\nu_1,\ldots,M_r+2\nu_r)$ for $\gamma=\otimes_a\gamma_a^{\nu_a}\in\Gamma_{\rm mir}$. Then the following quartet of states defines a B-type orientifold of Gepner model,

$$|\mathcal{C}^{B,\pi,\rho}_{\mathbf{M}}\rangle_{\rm NSNS+},\quad |\mathcal{C}^{B,\pi,\rho}_{\mathbf{M+2}}\rangle_{\rm NSNS-},\quad |\mathcal{C}^{B,\pi,\tilde\rho}_{\mathbf{M-1}}\rangle_{\rm RR+},\quad |\mathcal{C}^{B,\pi,\tilde\rho}_{\mathbf{M+1}}\rangle_{\rm RR-},$$
$$\tilde\rho(\vec\nu)\equiv\rho(\vec\nu)\exp\left(-\textstyle\sum_{a=1}^r\frac{i\pi\nu_a}{k_a+2}\right).$$

Here ρ is a character of Γ_O, whereas $\tilde\rho(\vec\nu)$ is anti-periodic in any of $\nu_a\to\nu_a+k_a+2$.

The label $\mathbf{M}$ is highly redundant because it has meanings only up to shifts by $\Gamma_{\rm mir}$. There is also a global $\mathbb{Z}_H$ symmetry of the mirror Gepner model that relates orientifolds with different $\mathbf{M}$.

Let us discuss the properties of the characters ρ of the group Γ_O in some detail. By definition, ρ is a character of the group $\Gamma_{\rm mir}$ that takes trivial value on the subgroup $\Gamma_{\rm mir}\Gamma^\pi_{\rm mir}$. The elements of this subgroup are given by $\vec\nu$ satisfying

$$\text{(i)}\ \sum_{a=1}^r\frac{\nu_a}{k_a+2}\in\mathbb{Z},$$
$$\text{(ii)}\ \nu_a=\nu_{\pi(a)},$$
(iii) ν_a is even for all a labelled by even k_a and fixed by π.

Characters of $\Gamma_{\rm mir}$ taking trivial value at such $\vec\nu$'s are given by

$$\rho(\vec\nu)\;=\;\prod_{c\ (\pi_c=(a_cb_c))}e^{-\frac{2\pi ir_c}{k_c+2}(\nu_{a_c}-\nu_{b_c})}\cdot\prod_{c\ (\|\pi_c\|=1,k_c={\rm even})}\epsilon_c^{\nu_c}.\tag{5.42}$$

Here $r_c\in\mathbb{Z}_{k_c+2}$ is associated to the cycle π_c of length 2 labelled by k_c, and the sign ϵ_c is associated to the length-one cycle π_c labelled by an even level k_c. Sometimes the conditions (i)–(iii) accidentally imply that some more ν_a have to be even, and ρ depends upon additional $\pm$ signs (see the Example 2 below). Finally, some of the parameters $(r_c,\epsilon_{c'})$ are redundant because of the equivalence $\rho(\vec\nu)\simeq\rho(\vec\nu)\exp\left(\sum_a\frac{2\pi i\nu_a}{k_a+2}\right)$ that follows from (i).

Recall that $|\mathcal{C}^{B,\pi,\rho}_{\mathbf{M}}\rangle$ is constructed by summing the crosscap states sitting in different twisted sectors. In the formula (5.42) for characters, the parameters r_c assign different weights to different twisted sectors so that they express the dressings by quantum symmetry of the mirror Gepner model. Such symmetry are known to map to the global symmetry of the original Gepner model. In other words, r_c's can be absorbed by a suitable redefinition of the LG fields $X_1,\ldots,X_r$. On the other hand, different signs ϵ_c give physically inequivalent orientifolds since they cannot be gauged away in such a way. In particular, the tension and supersymmetry phase φ of orientifolds do depend on ϵ's in a non-trivial manner.

Example 1: (55555) There are three inequivalent choices of permutations, π = id, $(12),(12)(34)$. For each choice of π there is a unique choice for $\mathbf{M}$ up to shifts by $\Gamma_{\rm mir}$ and the global $\mathbb{Z}_5$ symmetry of the mirror Gepner model. The tension is given by $c_{\rm NS}$ up to some positive proportionality constant, and the supersymmetry phase φ is given by (5.35).

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The group $\Gamma_{\rm mir}/(\Gamma_{\rm mir}\Gamma^{\pi}_{\rm mir})$ and the allowed character ρ for various choices of permutation are given by the following table (we denote $\omega_n \equiv \exp\frac{2\pi i}{n}$),

π	$\Gamma_{\rm mir}/(\Gamma_{\rm mir}\Gamma^{\pi}_{\rm mir})$	$\rho(\vec{\nu})$	
id	$\{1\}$	1	
(12)	$\mathbb{Z}_5$	$\omega_5^{-r(\nu_1-\nu_2)}$	$r \in \mathbb{Z}_5$
(12)(34)	$(\mathbb{Z}_5)^2$	$\omega_5^{-r(\nu_1-\nu_2)-r'(\nu_3-\nu_4)}$	$r, r' \in \mathbb{Z}_5$

The orientifolds labelled by different r, r' are related by quantum symmetries, so they are physically equivalent. We thus found three inequivalent B-type orientifolds of this model corresponding to three different choices of π.

Example 2: (88444) The orbifold group is $\Gamma_{\rm mir} = \mathbb{Z}_8 \times (\mathbb{Z}_4)^3$, and there are four inequivalent choices for the permutation, $\pi = {\rm id}, (12), (34)$ and $(12)(34)$. For each choice of π there are two inequivalent values for the label $\mathbf{M}$ up to shifts by $\Gamma_{\rm mir}$ and global symmetry of the mirror model,

$$\mathbf{M} = (00000) \quad \text{or} \quad (20000).$$

The orientifolds are also labelled by the character of the group $\Gamma_O \equiv \Gamma_{\rm mir}/(\Gamma_{\rm mir}\Gamma^{\pi}_{\rm mir})$. We determine the general form of the character following the argument given above ($\omega_n \equiv \exp\frac{2\pi i}{n}$),

$$\begin{array}{lrcll}
\pi = {\rm id} & \rho_{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4,\epsilon_5}(\vec{\nu}) & = & \epsilon_1^{\nu_1}\epsilon_2^{\nu_2}\epsilon_3^{\nu_3}\epsilon_4^{\nu_4}\epsilon_5^{\nu_5} & \simeq \rho_{-\epsilon_1,-\epsilon_2,\epsilon_3,\epsilon_4,\epsilon_5}(\vec{\nu}), \\
\pi = (12) & \rho_{r,\epsilon_1,\epsilon_3,\epsilon_4,\epsilon_5}(\vec{\nu}) & = & \omega_8^{-r(\nu_1-\nu_2)}\epsilon_1^{\nu_1}\epsilon_3^{\nu_3}\epsilon_4^{\nu_4}\epsilon_5^{\nu_5} & \simeq \rho_{r+2,-\epsilon_1,-\epsilon_3,-\epsilon_4,-\epsilon_5}(\vec{\nu}), \\
\pi = (34) & \rho_{r,\epsilon_1,\epsilon_2,\epsilon_5}(\vec{\nu}) & = & \omega_4^{-r(\nu_3-\nu_4)}\epsilon_1^{\nu_1}\epsilon_2^{\nu_2}\epsilon_5^{\nu_5} & \simeq \rho_{r,-\epsilon_1,-\epsilon_2,\epsilon_5}(\vec{\nu}), \\
\pi = (12)(34) & \rho_{r,r',\epsilon_1,\epsilon_5}(\vec{\nu}) & = & \omega_8^{-r(\nu_1-\nu_2)}\omega_4^{-r'(\nu_3-\nu_4)}\epsilon_1^{\nu_1}\epsilon_5^{\nu_5} & \simeq \rho_{r+2,r'+2,-\epsilon_1,-\epsilon_5}(\vec{\nu}).
\end{array} \tag{5.43}$$

In the second and fourth cases above, we have one more $\pm$ sign as compared to the formula (5.42) due to the accidental effect explained there. For example, for $\pi = (12)$ the elements of $\Gamma\Gamma^{\pi}$ are given by those $\vec{\nu}$ satisfying

$$\nu_1 + \nu_2 + 2(\nu_3 + \nu_4 + \nu_5) \in 8\mathbb{Z}, \qquad \nu_1 = \nu_2, \qquad \nu_{3,4,5} \in 2\mathbb{Z}. \tag{5.44}$$

These conditions imply that ν_1 is also even, so we get an additional parameter ϵ_1 in the second line of (5.43). Although ν_2 is also even, we do not introduce $\epsilon_2^{\nu_2}$ because $\epsilon_2^{\nu_2} = \omega_8^{4(\nu_1-\nu_2)}\epsilon_1^{\nu_2}$.

Let us compute the tension of the orientifolds we have listed, focusing on the dependence on ϵ-labels. We use various symmetry to set $\mathbf{M} = (00000)$ or (20000), $\epsilon_1 = 1$ and

$r, r' = 0$. The tension of various orientifolds then becomes,

$$\begin{aligned}
T(\mathcal{C}^{B,\mathrm{id},+\epsilon_2\epsilon_3\epsilon_4\epsilon_5}_{\mathbf{M}=(00000)}) &= c_{\mathrm{NS}} T_0^{(\mathrm{id})} (\cos\tfrac{\pi}{8})^{4-\alpha}(-i\sin\tfrac{\pi}{8})^{\alpha},\\
T(\mathcal{C}^{B,\mathrm{id},+\epsilon_2\epsilon_3\epsilon_4\epsilon_5}_{\mathbf{M}=(20000)}) &= c_{\mathrm{NS}} T_0^{(\mathrm{id})} (\cos\tfrac{\pi}{8})^{3-\alpha}(-i\sin\tfrac{\pi}{8})^{\alpha}\delta_{\epsilon_2,+},\\
T(\mathcal{C}^{B,(12),+\epsilon_3\epsilon_4\epsilon_5}_{\mathbf{M}=(00000)}) &= c_{\mathrm{NS}} T_0^{(12)} (\cos\tfrac{\pi}{8})^{3-\alpha}(-i\sin\tfrac{\pi}{8})^{\alpha},\\
T(\mathcal{C}^{B,(12),+\epsilon_3\epsilon_4\epsilon_5}_{\mathbf{M}=(20000)}) &= 0,\\
T(\mathcal{C}^{B,(34),+\epsilon_2\epsilon_5}_{\mathbf{M}=(00000)}) &= c_{\mathrm{NS}} T_0^{(34)} (\cos\tfrac{\pi}{8})^{2-\alpha}(-i\sin\tfrac{\pi}{8})^{\alpha},\\
T(\mathcal{C}^{B,(34),+\epsilon_2\epsilon_5}_{\mathbf{M}=(20000)}) &= c_{\mathrm{NS}} T_0^{(34)} (\cos\tfrac{\pi}{8})^{1-\alpha}(-i\sin\tfrac{\pi}{8})^{\alpha}\delta_{\epsilon_2,+},\\
T(\mathcal{C}^{B,(12)(34),+\epsilon_5}_{\mathbf{M}=(00000)}) &= c_{\mathrm{NS}} T_0^{(12)(34)} (\cos\tfrac{\pi}{8})^{1-\alpha}(-i\sin\tfrac{\pi}{8})^{\alpha},\\
T(\mathcal{C}^{B,(12)(34),+\epsilon_5}_{\mathbf{M}=(20000)}) &= 0.
\end{aligned} \tag{5.45}$$

Here α denotes the number of ϵ_a's taking $(-)$ sign, and T_0^{π} are some positive definite constants. In order to make the tension real, one therefore have to put $c_{\mathrm{NS}} = \pm i^{\alpha}$. A useful relation is $(-)^{\alpha} = c_{\mathrm{NS}}^2 = \rho(\vec{\nu} = \vec{1})$.

6. Some string theory problems

In this section we wish to study some more properties of permutation branes and orientifolds in Gepner models. One important problem is to find out the spectrum of massless open string modes. Here we will restrict our attention to the gauge fields on D-brane worldvolumes and study what gauge group is realized on coincident D-branes, by analyzing the action of parity on D-branes and open strings. Another important problem is to solve the tadpole cancellation condition and find supersymmetric tadpole-free configurations. The tadpole cancellation in general simply amounts to the cancellation of D-brane charges against the charge of orientifold. It becomes more and more difficult to solve it as the dimension of charge lattice gets larger. For type IIA case, we will analyze in a similar way as in [12] and find a few solutions involving permutation orientifolds using the simple relations between the charges of D-branes and orientifolds in minimal models. For type IIB we see that the charges of D-branes and orientifolds are summarized into simple polynomials as was discussed in [4, 16, 22, 43].

6.1 Parity action on D-branes

We would like to find out here the action of various orientifolds of Gepner model on D-branes from Möbius strips amplitudes. We begin with the Möbius strips in the product of r minimal models,

$$\begin{aligned}
{}_{\mathrm{NSNS}+}\langle \mathcal{B}^{\sigma}_{\mathbf{L},\mathbf{M}}|q^H|\mathcal{C}^{\pi}_{2\mathbf{m}}\rangle_{\mathrm{NSNS}\pm} &= {}_{\mathrm{NSNS}\mp}\langle \mathcal{C}^{\pi}_{2\mathbf{m}}|q^H|\mathcal{B}^{\sigma'}_{\mathbf{L},\mathbf{M}'}\rangle_{\mathrm{NSNS}+},\\
{}_{\mathrm{RR}-}\langle \mathcal{B}^{\sigma}_{\mathbf{L},\mathbf{M}}|q^H|\mathcal{C}^{\pi}_{2\mathbf{m}-\mathbf{1}}\rangle_{\mathrm{RR}\pm} &= {}_{\mathrm{RR}\mp}\langle \mathcal{C}^{\pi}_{2\mathbf{m}-\mathbf{1}}|q^H|\mathcal{B}^{\sigma'}_{\mathbf{L},\mathbf{M}''}\rangle_{\mathrm{RR}+}(-)^{r+|\sigma|+|\pi|}.
\end{aligned} \tag{6.1}$$

Here $\sigma' \equiv \pi\sigma^{-1}\pi$, and $\mathbf{M}', \mathbf{M}''$ have the following components,

$$M'_c = \sum_{a\in\sigma'_c} 2m_a - M_c, \qquad M''_c = \sum_{a\in\sigma'_c}(2m_a - 1) - M_c.$$

The minus signs in the second line come from the coefficients $\beta_{M,\mathrm{Y}}, \alpha_{\mathrm{Y}}$ in (4.11), (4.27) and (4.32). By taking the sum over orbits of the orbifold groups $\mathbb{Z}_H$ or Γ_{mir} we find the Möbius strip amplitudes for A-type crosscaps,

$$\begin{aligned} {}_{\mathrm{NSNS}+}\langle \mathcal{B}^{A,\sigma}_{\mathbf{L},\mathbf{M}}| q^H |\mathcal{C}^{A,\pi,\epsilon}_{\mathbf{2m}}\rangle_{\mathrm{NSNS}\pm} &= {}_{\mathrm{NSNS}\mp}\langle \mathcal{C}^{A,\pi,\epsilon}_{\mathbf{2m+2}}| q^H |\mathcal{B}^{A,\sigma'}_{\mathbf{L},\mathbf{M}'}\rangle_{\mathrm{NSNS}+} \cdot \epsilon c^2_{\mathrm{NS}} \\ {}_{\mathrm{RR}-}\langle \mathcal{B}^{A,\sigma}_{\mathbf{L},\mathbf{M}}| q^H |\mathcal{C}^{A,\pi,\tilde\epsilon}_{\mathbf{2m-1}}\rangle_{\mathrm{RR}\pm} &= {}_{\mathrm{RR}\mp}\langle \mathcal{C}^{A,\pi,\tilde\epsilon}_{\mathbf{2m+1}}| q^H |\mathcal{B}^{A,\sigma'}_{\mathbf{L},\mathbf{M}''}\rangle_{\mathrm{RR}+} \cdot \tilde\epsilon (-)^{r+|\sigma|+|\pi|}. \end{aligned} \tag{6.2}$$

Here $\tilde\epsilon$ was defined in (5.34). Recalling that c_{NS} was determined so that $\epsilon c^2_{\mathrm{NS}} = 1$, we conclude

$$\mathcal{C}^{A,\pi,\epsilon}_{\mathbf{2m}} \ : \ \mathcal{B}^{A,\sigma}_{\mathbf{L},\mathbf{M}} \ \mapsto \ \mathcal{B}^{A,\sigma'}_{\mathbf{L},\mathbf{M}'} \cdot (-)^{\frac{1}{2}(r+D)+|\pi|+|\sigma|}\epsilon, \tag{6.3}$$

where $\sigma, \mathbf{M}'$ are defined above and the $\pm$ sign distinguishes the brane and antibranes, i.e. $-\mathcal{B}$ denotes the antibrane of $\mathcal{B}$. The rule for B-type orientifolds is similar,

$$\mathcal{C}^{B,\pi,\rho}_{\mathbf{2m}} \ : \ \mathcal{B}^{B,\sigma}_{\mathbf{L},\mathbf{M}} \ \mapsto \ \mathcal{B}^{B,\sigma'}_{\mathbf{L},\mathbf{M}'} \cdot (-)^{\frac{1}{2}(r+D)+|\pi|+|\sigma|}\rho(\vec\nu = \vec 1), \tag{6.4}$$

where $\rho(\vec\nu)$ specifies a character of Γ_O (5.41). So the condition for a brane to be parity-invariant is $\pi\sigma^{-1}\pi = \sigma$ and $(\mathbf{L},\mathbf{M}) = (\mathbf{L},\mathbf{M}')$ up to shifts of $\mathbf{M}$ by orbifold elements and an even or odd times of brane identification $\mathcal{F}_c$ (5.11) depending on the sign in the above formulae.

For later use, we study the pairs of brane and orientifold satisfying the condition $(\mathbf{L},\mathbf{M}) = (\mathbf{L},\mathbf{M}')$ up to $\mathcal{F}_c$ by decomposing into blocks.

Condition for Parity Invariant Branes (PIB) 2. *If a brane $\mathcal{B}^{\sigma}_{\mathbf{L},\mathbf{M}}$ in the theory $\otimes_a M(k_a)$ is invariant under $\mathcal{C}^{\pi}_{\bar{\mathbf{M}}}$ ($M_a = M_{\pi(a)}$), the pair (π,σ) decomposes into the blocks listed in* PIB 1. *For each block of type* (1)–(3) *of* PIB 1,

$$\begin{aligned} &(1)\ \sigma_c = (a_1 a_2 \cdots a_{2n+1}),\ \pi = (a_1 a_{2n+1})(a_2 a_{2n})\cdots(a_n a_{n+2}),\\ &(2)\ \sigma_c = (a_1 a_2 \cdots a_{2n}),\quad \pi = (a_2 a_{2n+1})(a_2 a_{2n})\cdots(a_n a_{n+2}),\\ &(3)\ \sigma_c = (a_1 a_2 \cdots a_{2n}),\quad \pi = \quad (a_1 a_{2n})(a_2 a_{2n})\cdots(a_n a_{n+1}), \end{aligned}$$

the labels (L_c, M_c) have to satisfy

$$\begin{aligned} &\text{(I)}\ \ L_c = \text{any},\ M_c = \tfrac{1}{2}\bar M^{(\mathrm{tot})}_c \ \text{ or } \ \tfrac{1}{2}\bar M^{(\mathrm{tot})}_c + k_c + 2,\\ \text{or}\ &\text{(II)}\ L_c = \tfrac{k}{2},\quad M_c = \tfrac{1}{2}\bar M^{(\mathrm{tot})}_c \pm \tfrac{k_c+2}{2}, \end{aligned} \qquad \left(\bar M^{(\mathrm{tot})}_c \equiv \sum_{a\in\sigma_c} \bar M_a\right)$$

and for each block of type (4) *of the list,*

$$(4)\quad \sigma_c\sigma_{c'} = (a_1\cdots a_n)(a'_1\cdots a'_n),\quad \pi = (a_1 a'_n)(a_2 a'_{n-1})\cdots(a_n a'_1),$$

the labels $(L_c, M_c), (L_{c'}, M_{c'})$ have to satisfy

$$\begin{aligned} &\text{(III)}\ L_c = L_{c'},\qquad M_c + M_{c'} = \bar M^{(\mathrm{tot})},\\ \text{or}\ \ &\text{(IV)}\ L_c + L_{c'} = k,\ M_c + M_{c'} = \bar M^{(\mathrm{tot})} + k_c + 2. \end{aligned} \quad \left(\bar M^{(\mathrm{tot})} \equiv \sum_{a\in\sigma_c}\bar M_a \equiv \sum_{a\in\sigma_{c'}}\bar M_a\right)$$

Thus the pair $(\mathcal{B}^{\sigma}_{\mathbf{L},\mathbf{M}}, \mathcal{C}^{\pi}_{\bar{\mathbf{M}}})$ decomposes into eight different kinds of blocks,

$$(1)_{\mathrm{I}},\ \ (1)_{\mathrm{II}},\ \ (2)_{\mathrm{I}},\ \ (2)_{\mathrm{II}},\ \ (3)_{\mathrm{I}},\ \ (3)_{\mathrm{II}},\ \ (4)_{\mathrm{III}},\ \ (4)_{\mathrm{IV}}.$$

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Parity and supersymmetry. The action of parity on D-branes obtained above is such that the parity reversal of a supersymmetric configuration is again supersymmetric. Namely, if $|\mathcal{B}\rangle$ preserves the same supersymmetry as $|\mathcal{C}\rangle$, so does $P_{\mathcal{C}}|\mathcal{B}\rangle$. To see this, recall the supersymmetry phases for A-type branes and crosscaps,

$$\begin{aligned} \exp i\pi\varphi(\mathcal{B}^{A,\sigma}_{\mathbf{L},\mathbf{M}}) &= \exp i\pi\left(\sum_{c=1}^{[\sigma]}\frac{M_c}{k_c+2}-\frac{\|\sigma_c\|-1}{2}\right), \\ \exp i\pi\varphi(\mathcal{C}^{A,\pi,\epsilon}_{\mathbf{M}}) &= \frac{c_{\mathrm{R}}}{c_{\mathrm{NS}}}\exp i\pi\left(\sum_{a=1}^{r}\frac{M_a-1}{2k_a+4}+\frac{|\pi|+r}{2}\right), \end{aligned} \tag{6.5}$$

where $\|\sigma_c\|$ denotes the length of the c-th cycle of σ, and $|\pi|$ denotes the number of cycles of length 2 in π. Similar expressions hold also for B-types. Also, recall that $c_{\mathrm{R}}=\pm 1$, and that c_{NS} is determined from the group character as follows,

$$c_{\mathrm{NS}}^2\epsilon = 1 \quad \text{(A-type)} \ ; \quad c_{\mathrm{NS}}^2\rho(\vec{\nu}=\vec{1}) = 1 \quad \text{(B-type)}. \tag{6.6}$$

Combining these together with (6.3) or (6.4) one can show that, for any pair of an orientifold $\mathcal{C}$ and a D-brane $\mathcal{B}$,

$$\varphi(P_{\mathcal{C}}\mathcal{B}) \ = \ 2\varphi(\mathcal{C})-\varphi(\mathcal{B}) \pmod 2. \tag{6.7}$$

The formulae (6.3) and (6.4) determine the action of orientifolds on all the *long-orbit branes*, or branes with trivial untwisted stabilizer group $\mathcal{U}$. We need some more work to find out the action of orientifolds on *short-orbit branes* which have non-trivial $\mathcal{U}$ and are therefore labelled by additional label specifying a character of $\mathcal{U}$.

6.1.1 Parity action on short-orbit A-branes

Short-orbit A-branes are made from permutation boundary states in twisted sectors, $|\mathcal{B}^{(1\cdots N)}_{L,M}\rangle^{(\mu)}$ and $|\widetilde{\mathcal{B}}^{(1\cdots N)}_{k/2,M}\rangle^{(\mu)}$, in the product of N identical minimal models $M(k)^N$ defined in (5.14) and (5.15). They satisfy the basic transformation laws (here $\omega\equiv e^{\frac{2\pi i}{k+2}}$)

$$\begin{aligned} |\mathcal{B}^{(2\cdots N1)}_{L,M}\rangle^{(\mu)}_{\mathrm{NSNS}\pm} &= +|\mathcal{B}^{(12\cdots N)}_{L,M}\rangle^{(\mu)}_{\mathrm{NSNS}\pm}\omega^{M\mu}, \\ |\widetilde{\mathcal{B}}^{(2\cdots N1)}_{k/2,M}\rangle^{(\mu)}_{\mathrm{NSNS}\pm} &= \mp|\widetilde{\mathcal{B}}^{(12\cdots N)}_{k/2,M}\rangle^{(\mu)}_{\mathrm{NSNS}\pm}\omega^{M\mu}, \end{aligned} \tag{6.8}$$

$$\begin{aligned} \gamma_a|\mathcal{B}^{(12\cdots N)}_{L,M}\rangle^{(\mu)}_{\mathrm{Y}} &= +|\mathcal{B}^{(12\cdots N)}_{L,M+2}\rangle^{(\mu)}_{\mathrm{Y}}\omega^{\mu(2a-1)}, \\ \gamma_a|\widetilde{\mathcal{B}}^{(12\cdots N)}_{k/2,M}\rangle^{(\mu)}_{\mathrm{Y}} &= -|\widetilde{\mathcal{B}}^{(12\cdots N)}_{k/2,M+2}\rangle^{(\mu)}_{\mathrm{Y}}\omega^{\mu(2a-1)}. \end{aligned} \tag{6.9}$$

We study the action of NS parity $\mathcal{C}^{\pi}_{\bar{\mathbf{M}}}$ on these boundary states. It maps the σ-permuted boundary states to σ'-permuted boundary states, where

$$\sigma=(1\,2\cdots N) \quad\Longrightarrow\quad \sigma'=\pi\sigma^{-1}\pi=(\pi(N)\,\pi(N-1)\cdots\pi(1)). \tag{6.10}$$

The NS parity acts on Ishibashi states as

$$\begin{aligned} (-)^{F_L}P^{\pi}_{\bar{\mathbf{M}}}\,|\mathcal{B}^{\sigma};l,m\rangle\!\rangle^{(\mu)}_{\mathrm{NSNS}\pm} &= \otimes_a\gamma_a^{\bar{M}_a/2}\cdot|\mathcal{B}^{\sigma'};l,-m\rangle\!\rangle^{(\mu)}_{\mathrm{NSNS}\pm}, \\ (-)^{F_L}P^{\pi}_{\bar{\mathbf{M}}}\,|\widetilde{\mathcal{B}}^{\sigma};l,m\rangle\!\rangle^{(\mu)}_{\mathrm{NSNS}\pm} &= \otimes_a\gamma_a^{\bar{M}_a/2}\cdot|\widetilde{\mathcal{B}}^{\sigma'};l,k+2-m\rangle\!\rangle^{(\mu)}_{\mathrm{NSNS}\pm}\cdot(\pm i). \end{aligned} \tag{6.11}$$

Therefore the boundary states are transformed as,

$$\begin{aligned}(-)^{F_L} P^{\pi}_{\bar{\mathbf{M}}} |\mathcal{B}^{\sigma}_{L,M}\rangle^{(\mu)}_{\mathrm{NSNS}\pm} &= \otimes_a \gamma_a^{\bar{M}_a/2} \cdot |\mathcal{B}^{\sigma'}_{L,-M}\rangle^{(\mu)}_{\mathrm{NSNS}\pm}, \\ (-)^{F_L} P^{\pi}_{\bar{\mathbf{M}}} |\widetilde{\mathcal{B}}^{\sigma}_{L,M}\rangle^{(\mu)}_{\mathrm{NSNS}\pm} &= \otimes_a \gamma_a^{\bar{M}_a/2} \cdot |\widetilde{\mathcal{B}}^{\sigma'}_{L,-M}\rangle^{(\mu)}_{\mathrm{NSNS}\pm} \cdot (\mp i). \end{aligned} \tag{6.12}$$

The above formula can be directly applied to the parity action on short-orbit A-branes in Gepner models. A general permutation A-brane with stabilizer group $\mathbb{Z}_{H'}$ ($H' = H/h$) takes the form (5.16),

$$|\mathcal{B}^{A,\sigma,\rho}_{\mathbf{L},\mathbf{M}}\rangle_{\mathrm{Y}} = \frac{1}{\sqrt{H}} \sum_{\nu\in\mathbb{Z}_h} \sum_{\mu\in\mathbb{Z}_{H'}} \gamma^{\nu}_{(A)} \bigotimes_{c=1}^{[\sigma]} |\mathcal{B}^{\sigma_c}_{L_c,M_c}\rangle^{(\mu h)}_{\mathrm{Y}} \exp\left(\frac{2\pi i \mu h \rho}{H}\right). \tag{6.13}$$

The orientifolds $\mathcal{C}^{A,\pi,\epsilon}_{\bar{\mathbf{M}}}$ maps the brane $|\mathcal{B}^{A,\sigma,\rho}_{\mathbf{L},\mathbf{M}}\rangle$ to $\otimes_a \gamma_a^{\bar{M}_a/2} \cdot |\mathcal{B}^{A,\sigma',\rho'}_{\mathbf{L},-\mathbf{M}}\rangle$. The permutations σ and σ' are related cycle by cycle as follows,

$$\sigma_c = (a_1 \cdots a_n) \iff \sigma'_c = (\pi(a_n)\cdots\pi(a_1)). \tag{6.14}$$

The mod-H' integer ρ gets shifted according to the following rules:

1. ρ gets shifted by $\frac{H}{2} = \frac{hH'}{2}$ if H is even and the orientifold has $\epsilon = (-)$.
2. ρ gets shifted by $\frac{nH'}{2}$ if the boundary state in $\gamma^h_{(A)}$-twisted sector contains $2n$ tilded boundary states.

As an application, let us find out the condition for an A-brane $\mathcal{B}^{A,\sigma,\rho}_{\mathbf{L},\mathbf{M}}$ to be invariant under the A-type orientifold $\mathcal{C}^{A,\pi,\epsilon}_{\bar{\mathbf{M}}}$. For simplicity, we assume their labels are chosen in such a way that the pair $(\mathcal{B}^{\sigma}_{\mathbf{L},\mathbf{M}}, \mathcal{C}^{\pi}_{\bar{\mathbf{M}}})$ satisfy the condition PIB 2. The problem is then how the label ρ is transformed under the parity. Besides the possible shifts of ρ listed above, it gets shifted when we use the formula (6.8), (6.9) or the identification $\mathcal{F}_c$ to transform the labels $(\pi\sigma^{-1}\pi, \mathbf{L}, \mathbf{M}')$ into $(\sigma, \mathbf{L}, \mathbf{M})$. A detailed analysis shows

3. ρ gets shifted by $\frac{nH'}{2}$ if the boundary state in $\gamma^h_{(A)}$-twisted sector contains n tilded boundary states of type $(1)_{\mathrm{II}}, (2)_{\mathrm{II}}$ or $(3)_{\mathrm{II}}$.
4. ρ gets shifted by $(1+\frac{\bar{M}_{a_1}}{2})\frac{H'}{2}$ or $(h+1+\frac{\bar{M}_{a_1}}{2})\frac{H'}{2}$ if the boundary state in $\gamma^h_{(A)}$-twisted sector contains a tilded boundary states of type $(2)_{\mathrm{I}}$ or $(2)_{\mathrm{II}}$.

In any case, the action of orientifold on ρ of the brane $\mathcal{B}^{A,\sigma,\rho}_{\mathbf{L},\mathbf{M}}$ is at most a half period shift, and it only occurs when $\mathbf{L}$ is special so that the tilded boundary states are involved in its construction. The parity action on the label ρ is thus determined from the expression of boundary state in $\gamma^h_{(A)}$-twisted sector. Whether ρ is invariant or shifted by half-period is determined by the following sign (where the notation should be obvious from the above explanation),

$$\lambda \equiv \epsilon^h (-)^{\frac{1}{2}\sharp(\tilde{B}) - \sharp(\tilde{B}_{(\mathrm{II})})} \prod_{\tilde{B}:(2)_{\mathrm{I}}} (-)^{\frac{\bar{M}_{a_1}}{2}+1} \prod_{\tilde{B}:(2)_{\mathrm{II}}} (-)^{\frac{\bar{M}_{a_1}}{2}+1+h}. \tag{6.15}$$

6.1.2 Parity action on short-orbit B-branes

B-branes in Gepner model with nontrivial untwisted stabilizer $\mathcal{U}$ are made of permutation boundary states $|\mathcal{B}^{\sigma,\rho}_{L,M}\rangle$ defined at (5.19), with $\sigma = (12\cdots N)$ a cycle of even length. Each of $|\mathcal{B}^{\sigma,\rho}_{L,M}\rangle$ contributes a factor of $\mathbb{Z}_{k+2}$ or $\mathbb{Z}_{(k+2)/2}$ to $\mathcal{U}$, depending on whether L is generic or coincides with $k/2$. For the D-branes whose untwisted stabilizer contain the generator $\check{\gamma}$ of (5.27), we construct the boundary states in $\check{\gamma}$-twisted sector using $|\widetilde{\mathcal{B}}^{\sigma}_{k/2,M}\rangle$ and $|\mathcal{B}^{\sigma,\rho,-}_{k/2,M}\rangle$ defined at (5.28) and (5.31).

The σ-permuted short orbit B-branes are therefore labelled by the half integers (ρ_c), and also by a sign ε if $\mathcal{U}$ contains the element $\check{\gamma}$. Each ρ_c is associated to an even-length cycle σ_c, and has period k_c+2 or $(k_c+2)/2$ depending on whether L_c is generic or not. The sign ε appears in the expression for boundary states as follows,

$$\begin{aligned}|\mathcal{B}^{B,\sigma,(\rho,\varepsilon)}_{\mathbf{L},\mathbf{M}}\rangle \sim & \sum_{\otimes\gamma_a^{\nu_a}\in\Gamma_{\rm mir}/\mathcal{H}} \otimes_a\gamma_a^{\nu_a}\cdot \bigotimes^{(L_c\neq\frac{k_c}{2})}_{\rm odd}|\mathcal{B}^{\sigma_c}_{L_c,M_c}\rangle \bigotimes^{(L_c\neq\frac{k_c}{2})}_{\rm even}|\mathcal{B}^{\sigma_c,\rho_c}_{L_c,M_c}\rangle \\ & \times\left\{\bigotimes_{\rm odd}|\mathcal{B}^{\sigma_c}_{\frac{k_c}{2},M_c}\rangle\bigotimes_{\rm even}|\mathcal{B}^{\sigma_c,\rho_c,+}_{\frac{k_c}{2},M_c}\rangle+\varepsilon\bigotimes_{\rm odd}|\widetilde{\mathcal{B}}^{\sigma_c}_{\frac{k_c}{2},M_c}\rangle\bigotimes_{\rm even}|\mathcal{B}^{\sigma_c,\rho_c,-}_{\frac{k_c}{2},M_c}\rangle\right\}. \qquad (6.16)\end{aligned}$$

An example is the boundary state (5.32) for a B-brane in the (88444) model.

We wish to find out the action of various B-type (NSNS) parities on B-branes, in particular how the labels (ρ_c,ε) are transformed. We consider the parity $P^{B,\pi,\vec{r}}_{\bar{\mathbf{M}}}$ corresponding to a general B-type orientifold,

$$|\mathcal{C}^{B,\pi,\vec{r}}_{\bar{\mathbf{M}}}\rangle \;=\; \frac{1}{\sqrt{|\Gamma_{\rm mir}|}}\sum_{\otimes_a\gamma_a^{\nu_a}\in\Gamma_{\rm mir}}|\mathcal{C}^{\pi}_{\bar{\mathbf{M}}+2\vec{\nu}}\rangle\exp\left(-\textstyle\sum_a\frac{2\pi i r_a\nu_a}{k_a+2}\right).$$

Actually the transformation law of $\{\rho_c\}$ is obtained simply by applying the general formula (2.79), thanks to the fact that the boundary states in twisted sector is essentially unique unlike the case with A-branes (cf. equation (6.8)). To illustrate this, let us work out the condition on ρ-labels for a B-brane $\mathcal{B}^{B,\sigma,(\rho_c,\epsilon)}_{\mathbf{L},\mathbf{M}}$ to be invariant under the orientifold $\mathcal{C}^{B,\pi,\vec{r}}_{\bar{\mathbf{M}}}$.

Condition for Parity Invariant Branes (PIB) 3. *Take a pair $(\mathcal{B}^{\sigma}_{\mathbf{L},\mathbf{M}},\mathcal{C}^{\pi}_{\bar{\mathbf{M}}})$ satisfying the condition PIB 2. Then the B-type orientifold $\mathcal{C}^{B,\pi,\vec{r}}_{\bar{\mathbf{M}}}$ acts on the ρ-labels of the B-brane $\mathcal{B}^{B,\sigma,\rho}_{\mathbf{L},\mathbf{M}}$ in a non-trivial manner. By analyzing the condition of parity invariance on ρ block by block one finds the following:*

1. *the blocks of type* (1) *do not contain ρ-labels.*

2. *in a block of type* (2)*, the boundary state $\mathcal{B}^{\sigma_c,\rho_c}_{L_c,M_c}$ ($\sigma_c=(a_1a_2\cdots a_{2n})$) has the label ρ_c which transform under parity as*

$$\rho_c \;\mapsto\; \rho_c + r^{(\rm tot)},\quad r^{(\rm tot)}\equiv r_{a_1}-r_{a_2}+\cdots-r_{a_{2n}}.$$

It follows from the involutiveness of parity that $r^{(\rm tot)}=0$ or $\frac{k_c+2}{2}$ mod k_c+2. If the latter is the case L_c has to equal $k_c/2$, but there arise no condition on ρ_c.

3. in a block of type (3) *the parity transform the ρ-label as $\rho_c \mapsto -\rho_c - r^{(\mathrm{tot})}$, where $\rho_c, r^{(\mathrm{tot})}$ are defined similarly to the previous case. The parity invariance requires*

$$\begin{array}{rll} & (\mathrm{A}) : \rho_c = -\frac{1}{2}r^{(\mathrm{tot})} \mod \frac{k_c+2}{2}, & L_c = \text{any}, \\ \text{or} & (\mathrm{B}) : \rho_c = -\frac{1}{2}r^{(\mathrm{tot})} + \frac{k_c+2}{4} \mod \frac{k_c+2}{2}, & L_c = \frac{k_c}{2}. \end{array}$$

4. in a block of type (4)*, we take $\sigma_c \circ \sigma_{c'} = (a_1 \cdots a_{2n}) \circ (a'_1 \cdots a'_{2n})$ and consider the boundary state $\mathcal{B}^{\sigma_c,\rho_c}_{L_c,M_c} \otimes \mathcal{B}^{\sigma_{c'},\rho_{c'}}_{L_{c'},M_{c'}}$. The parity acts on the labels $\rho_c, \rho_{c'}$ as*

$$\begin{array}{ll} \rho_c \mapsto -r^{(\mathrm{tot})} - \rho_{c'}, & r^{(\mathrm{tot})} \equiv r_{a_1} - r_{a_2} + r_{a_3} \cdots - r_{a_{2n}} \\ \rho_{c'} \mapsto -r^{(\mathrm{tot})} - \rho_c, & \qquad\;\; = r_{a'_1} - r_{a'_2} + r_{a'_3} \cdots - r_{a'_{2n}}. \end{array}$$

The parity-invariant blocks of type $(4)_{\mathrm{III}}$ or $(4)_{\mathrm{IV}}$ have to satisfy

$$\begin{array}{l} (\mathrm{III}) : \rho_c + \rho_{c'} + r^{(\mathrm{tot})} = 0 \mod k_c + 2, \\ (\mathrm{IV}) : \rho_c + \rho_{c'} + r^{(\mathrm{tot})} = \frac{k_c+2}{2} \mod k_c + 2. \end{array}$$

The pair $(\mathcal{B}^{B,\sigma,\rho}_{\mathbf{L},\mathbf{M}}, \mathcal{C}^{B,\pi,\vec{r}}_{\bar{\mathbf{M}}})$ therefore decomposes into blocks of 10 different kinds,

$$(1)_{\mathrm{I}},\ (1)_{\mathrm{II}},\ (2)_{\mathrm{I}},\ (2)_{\mathrm{II}},\ (3)_{\mathrm{IA}},\ (3)_{\mathrm{IB}},\ (3)_{\mathrm{IIA}},\ (3)_{\mathrm{IIB}},\ (4)_{\mathrm{III}},\ (4)_{\mathrm{IV}}.$$

Parity action on ε. A naive application of the formula (2.79) does not work for determining the action of parity on ε because we have been making no distinction between $\gamma^{\frac{k+2}{2}}$-twisted sector and $\psi\gamma^{\frac{k+2}{2}}$-twisted sector of minimal models. Here we focus on short-orbit B-branes $\mathcal{B}^{B,\sigma,(\rho,\varepsilon)}_{\mathbf{L},\mathbf{M}}$ satisfying the condition PIB 3 discussed above and ask what is the relation between ε and ε' in the formula:

$$(-)^{F_L} P^{B,\pi,\vec{r}}_{\bar{\mathbf{M}}} \ :\ \mathcal{B}^{B,\sigma,(\rho,\varepsilon)}_{\mathbf{L},\mathbf{M}} \ \mapsto \mathcal{B}^{B,\sigma,(\rho,\varepsilon')}_{\mathbf{L},\mathbf{M}}.$$

The result is summarized as

$$\frac{\varepsilon'}{\varepsilon} \ = \ (-)^{\sharp(1)_{\mathrm{II}}+\sharp(2)_{\mathrm{II}}+\sharp(3)_{\mathrm{IIA}}+\sharp(3)_{\mathrm{IIB}}} \cdot (-)^{\sharp(3)_{\mathrm{IB}}+\sharp(3)_{\mathrm{IIB}}} \cdot \prod_{\sigma_c \text{ odd},\ L_c=k_c/2} (-i)\cdot(-)^{\sum_{a\in\sigma_c} r_a}, \quad (6.17)$$

where $\sharp(\cdots)$ counts the number of blocks of each type. The factors in the right hand side arise from the following reason. The first sign $(-)^{\sharp(1)_{\mathrm{II}}+\sharp(2)_{\mathrm{II}}+\sharp(3)_{\mathrm{IIA}}+\sharp(3)_{\mathrm{IIB}}}$ arises because the states $|\widetilde{\mathcal{B}}^{\sigma}_{k/2,M}\rangle$, $|\mathcal{B}^{\sigma,\rho,-}_{k/2,M}\rangle$ are odd under the shift $M \to M+k+2$. The second sign $(-)^{\sharp(3)_{\mathrm{IB}}+\sharp(3)_{\mathrm{IIB}}}$ is from the states $|\mathcal{B}^{\sigma,\rho,-}_{k/2,M}\rangle$ which are odd under the shift $\rho \to \rho + \frac{k+2}{2}$. The last factor arises from the odd-length cycles σ_c labelled by $L_c = k_c/2$. A $(-i)$ is due to the parity action

$$(-)^{F_L} P^{\pi}_{\bar{\mathbf{M}}} |\widetilde{B}^{\sigma}_{k/2,M}\rangle_{\mathrm{NSNS}\pm} \ = \ \mp i |\widetilde{B}^{\sigma}_{k/2,\bar{M}_{\mathrm{tot}}-M}\rangle_{\mathrm{NSNS}\pm}.$$

The r_a-dependent sign arises from the action of quantum symmetry labelled by $\vec{r}$ on states sitting in $(\eta_{a_1}\cdots\eta_{a_n})$-twisted sector.

6.2 Gauge group

If a brane $\mathcal{B}$ is invariant under the orientifold $\mathcal{C}$, then the corresponding Möbius strip amplitude shows a massless gauge boson running along the strip. The parity eigenvalue of the gauge boson determines whether the gauge group is O or Sp. We read off the eigenvalues of NS parities $(-)^{F_L}P_{\mathcal{C}}$ or $(-)^{F_R}P_{\mathcal{C}}$ for the orientifold $\mathcal{C}$ from the amplitudes

$$\mp i \;{}_{\rm NSNS+}\langle\mathcal{B}|q^H|\mathcal{C}\rangle_{\rm NSNS\pm} \;=\; \mp i \;{}_{\rm NSNS\mp}\langle\mathcal{C}|q^H|\mathcal{B}\rangle_{\rm NSNS+}.$$

We regard $\mp i$ as the value of NS parities for open string NS ground state. Since NS parities square to fermion number, it follows that the NS tachyon (and all the NS states that are projected out by GSO projection) has *odd* fermion number, and the remaining states have eigenvalues ± 1 of the NS parities. The gauge group is O or Sp depending on the gauge boson having eigenvalues -1 or 1 of NS parities.

We compute the eigenvalues of NS parities by decomposing the Möbius strip amplitudes into parts. The spacetime part of the amplitude reads

$$\mp i \;{}_{\rm NSNS+}^{\rm st}\langle\mathcal{B}|e^{-\pi H_c/4l}|\mathcal{C}\rangle^{\rm st}_{\rm NSNS\pm} \;\sim\; \mp i\cdot q^{-\frac{c_{\rm st}}{24}-\frac12}\{\widehat{\chi}_0(q)\mp i\widehat{\chi}_2(q)\}^d e^{\pm\frac{i\pi d}{4}} \qquad (q\equiv e^{-2\pi l})$$

where χ_s are characters of $\mathrm{U}(1)_2$ and the hat operation is defined in (2.10). The spacetime part therefore contributes $-e^{\frac{\pm i\pi d}{4}}$ to the eigenvalue of $(-)^{F_{L,R}}P_{\mathcal{C}}$ on gauge boson. The internal part, if the brane is parity invariant, can be studied by decomposing them into blocks as explained in section 2.5.1. Let us forget about the orbifolding for the moment and first consider Möbius strip of a single minimal model,

$$\begin{aligned}{}_{\rm NSNS+}\langle\mathcal{B}_{L,M}|e^{-\frac{\pi H_c}{4\ell}}|\mathcal{C}_{\bar M}\rangle_{\rm NSNS\pm} = \sum_{l=0}^{\min(L,k-L)} &\Big\{(-)^{l+L-\frac{\bar M}{2}}e^{\mp\frac{i\pi}{4}}\widehat{\chi}^{\rm NS\mp}_{2l,2M-\bar M}(q)\\ &+e^{\pm\frac{i\pi}{4}}\widehat{\chi}^{\rm NS\mp}_{k-2l,2M-\bar M-k-2}(q)\Big\},\end{aligned}\tag{6.18}$$

where $\widehat{\chi}^{\rm NS\pm}_{l,m}$ are linear combinations of hatted characters in minimal model,

$$\widehat{\chi}^{\rm NS\pm}_{l,m} \;\equiv\; \sigma_{lm0}\widehat{\chi}_{l,m,0}\pm i\sigma_{lm2}\widehat{\chi}_{l,m,2},\tag{6.19}$$

and $\sigma_{lms}=e^{i\pi\theta(l,m,s)}$ was defined at (4.3). From the coefficient of $\widehat{\chi}^{\rm NS\pm}_{0,0}$ one finds the value of NS parities on the ground state,

$$\begin{array}{lll} L=\text{any}, & M=\frac{\bar M}{2}\text{ or }\frac{\bar M}{2}+k+2 & \Rightarrow (-)^{F_{L,R}}P_{\bar M}=e^{\mp\frac{i\pi}{4}},\\ L=\frac{k}{2}, & M=\frac{\bar M}{2}\pm\frac{k+2}{2} & \Rightarrow (-)^{F_{L,R}}P_{\bar M}=e^{\pm\frac{i\pi}{4}}.\end{array}\tag{6.20}$$

We generalize this analysis to the pairs of a permutation brane $\mathcal{B}^\sigma_{\mathbf{L},\mathbf{M}}$ and orientifold $\mathcal{C}^\pi_{\bar{\mathbf{M}}}$ in tensor products of minimal models. We again assume $\bar M_a=\bar M_{\pi(a)}$ for simplicity. We decompose them into blocks satisfying the condition PIB 2 and compute the values of NS parities block by block.

NS Parity eigenvalue formula.

$$\begin{aligned} &(1)_{\rm I} \ : (-)^{F_L}P = e^{-\frac{i\pi}{4}}, && (3)_{\rm I} \ : (-)^{F_L}P = -i, \\ &(1)_{\rm II} : (-)^{F_L}P = e^{+\frac{i\pi}{4}}, && (3)_{\rm II} \ : (-)^{F_L}P = 1, \\ &(2)_{\rm I} \ : (-)^{F_L}P = 1, && (4)_{\rm III} : (-)^{F_L}P = 1, \\ &(2)_{\rm II} : (-)^{F_L}P = -i(-)^{\frac{\bar{M}_{a_1}}{2}}, && (4)_{\rm IV} : (-)^{F_L}P = 1. \end{aligned} \tag{6.21}$$

To determine the gauge group on D-branes in Gepner model, one has to combine the NS parity eigenvalue from all the blocks together with the overall coefficient of the crosscap $c_{\rm NS}$, and then sum over orbifold images.

Let us start with type IIA and consider a brane $\mathcal{B}^{A,\sigma,\rho}_{\mathbf{L},\mathbf{M}}$ invariant under the orientifold $C^{A,\pi,\epsilon}_{\bar{\mathbf{M}}}$. The Möbius strip amplitude is given by the sum over orbifold orbit,

$$\begin{aligned} {}_{\rm Y}\langle \mathcal{B}^{A,\sigma,\rho}_{\mathbf{L},\mathbf{M}}|q^H|\mathcal{C}^{A,\pi,\epsilon}_{\bar{\mathbf{M}}}\rangle_{\rm Y'} &= \frac{1}{|\mathcal{H}|}\sum_{\gamma\in\Gamma} {}_{\rm Y}\langle \mathcal{B}^{\sigma}_{\mathbf{L},\mathbf{M}}|q^H|\mathcal{C}^{\pi}_{\gamma(\bar{\mathbf{M}})}\rangle_{\rm Y'}\epsilon(\gamma)c_{\rm NS} \\ &\equiv \frac{1}{|\mathcal{H}|}\sum_{\gamma\in\Gamma}\mathcal{M}(\gamma), \end{aligned} \tag{6.22}$$

where $\epsilon(\gamma) \equiv \epsilon^{\nu}$ when $\gamma(\bar{\mathbf{M}}) = \bar{\mathbf{M}} + \mathbf{2}\nu$, and $\mathcal{H} \subset \Gamma$ is the stabilizer group of the brane. In the sum in the right hand side, there are $|\mathcal{H}|$ terms satisfying the condition PIB 2 and therefore contributing to the NS parity eigenvalue. However, for generic $\mathbf{L}$ the sum is trivial so that it simply removes the factor $1/|\mathcal{H}|$ in front. If $\mathbf{L}$ is such that the enhancement of the stabilizer group occurs, the sum boils down to an average of two terms with γ being identity or the generator $\gamma^h_{(A)}$ of the stabilizer group. Expanding $\mathcal{M}(\text{id})$ and $\mathcal{M}(\gamma^h)$ as power series in the loop-channel modular parameter, the coefficients of the leading term gives the eigenvalues of operators $(-)^{F_L}P$ and $(-)^{F_L}\gamma^h_{(A)}P$ on ground state. The value of $\gamma^h_{(A)}$ on open string ground state obtained in this way should coincide with λ at (6.15).

Let us next consider type IIB case and take a brane $\mathcal{B}^{B,\sigma,(\rho,\varepsilon)}_{\mathbf{L},\mathbf{M}}$ invariant under the orientifold $C^{B,\pi,\vec{r}}_{\bar{\mathbf{M}}}$. The parity eigenvalue of NS ground state on the brane can be computed by summing the Möbius strips $\mathcal{M}(\gamma)$ in the product of minimal models satisfying the condition PIB 3. When σ contains a cycle σ_c of even length, this involves summing $\mathcal{M}(\gamma)$ over orbits generated by the elements $\gamma_{\sigma_c} \in \mathcal{U}$ defined at (5.26). This not only enforces the condition PIB 3 on ρ_c but moreover projects out the terms containing blocks of type $(2)_{\rm II}, (3)_{\rm IB}$ and $(3)_{\rm IIA}$. The terms which survive this averaging are therefore those consisting only of the blocks

$$(1)_{\rm I}, (1)_{\rm II}, (2)_{\rm I}, (3)_{\rm IA}, (3)_{\rm IIB}, (4)_{\rm III}, (4)_{\rm IV}.$$

The non-trivial part of averaging thus amounts to the sum over $\gamma \in (\mathbb{Z}_2)^{p-1} \subset \Gamma_{\rm mir}$, where p is the number of odd-length cycles σ_c labelled by $L_c = \frac{k_c}{2}$ and $(\mathbb{Z}_2)^{p-1}$ is the group of even-order monomials of $\eta^{\rm (tot)}_c \equiv \prod_{a\in\sigma_c}\eta_a$. Including the spacetime part and other factors, the NS parity eigenvalue of gauge bosons finally becomes

$$\begin{aligned} (-)^{F_L}P &= -c_{\rm NS}(-i)^{\frac{1}{2}\{\sharp(1)_{\rm I}-\sharp(1)_{\rm II}-d\}+\sharp(3)_{\rm IA}} \times \\ &\times \mathrm{Re}\left(2^{-[p/2]}(-i)^{\sharp(1)_{\rm II}}\cdot \prod_{\sigma_c\ \text{odd},\, L_c=k_c/2}\left(1+i(-)^{\sum_{a\in\sigma_c}r_a}\right)\right). \end{aligned} \tag{6.23}$$

$\pi = \text{id},\ \varphi(\mathcal{C}) = 0$		
σ	$\varphi(\mathcal{B})$	$\widetilde{P}$
id	0	-1
(12)	$-\frac{1}{2}$	$-i$
(12)(34)	-1	$+1$

$\pi = (12),\ \varphi(\mathcal{C}) = \frac{1}{2}$		
σ	$\varphi(\mathcal{B})$	$\widetilde{P}$
id	0	$-i$
(12)	$-\frac{1}{2}$	-1
(34)	$-\frac{1}{2}$	$+1$
(12)(34)	-1	$-i$
(123)	-1	$-i$
(123)(45)	$-\frac{3}{2}$	$+1$

$\pi = (12)(34),\ \varphi(\mathcal{C}) = 1$		
σ	$\varphi(\mathcal{B})$	$\widetilde{P}$
id	0	$+1$
(12)	$-\frac{1}{2}$	$-i$
(345)	-1	$+1$
(12)(34)	-1	-1
(13)(24)	-1	$+1$
(1234)	$-\frac{3}{2}$	$-i$
(12)(345)	$-\frac{3}{2}$	$-i$
(13542)	-2	$+1$

Table 3: Parity eigenvalue of gauge boson on various D-branes of the model (55555).

6.2.1 Example 1: (55555)

Let us study the gauge group on A-branes $|\mathcal{B}^{A,\sigma}_{\mathbf{L},\mathbf{M}}\rangle$ in the model (55555) which are invariant under the orientifold $|\mathcal{C}^{A,\pi}_{\bar{\mathbf{M}}}\rangle$. We put $c_{\rm NS} = -1$ and set $\mathbf{M} = \bar{\mathbf{M}} = \mathbf{0}$ for simplicity. For each of the allowed σ's we compute the supersymmetry phase of the brane $|\mathcal{B}^{\sigma}_{\mathbf{L},\mathbf{0}}\rangle$ and the eigenvalue of corresponding NS parity $\widetilde{P}$ and summarize them in the table 3 below. Because H is odd, the parity eigenvalue are computed simply by multiplying the contributions from blocks.

When the eigenvalue of $\tilde{P}$ is pure imaginary, the gauge boson has $(-)^F = -1$ and is therefore GSO projected out. This is in consistency with that the brane $\mathcal{B}$ is mapped to its anti-brane under an orientifold $\mathcal{C}$ when $\varphi(\mathcal{B}) - \varphi(\mathcal{C}) = \frac{1}{2}$ (mod $\mathbb{Z}$), as the table shows.

Since nontrivial stabilizer group or summing over orbifold images do not affect the computation of parity eigenvalue, the analysis for B-type branes and orientifolds is essentially the same and the result summarized in table 3 applies also to B-types.

6.2.2 Example 2: (88444)

We take this model to discuss the gauge group on branes with special **L**-labels. We first present some type IIA examples:

- Consider a non-permuted brane $\mathcal{B}^{A,\sigma=\text{id}}_{\mathbf{L},\mathbf{M}}$ invariant under the orientifold $\mathcal{C}^{A,\pi=\text{id},+}_{\bar{\mathbf{M}}}$. When $c_{\rm NS} = -1$, the branes with generic $\mathbf{L}$ support $O(N)$ gauge group. If $L_1 = L_2 = 3$ the branes split into two short-orbit branes exchanged to each other by orientifold because λ of (6.15) takes -1, and the short-orbit brane supports a unitary gauge group.

- Consider a pair $(\mathcal{B}^{A,(12)(345)}_{\mathbf{L},\mathbf{M}},\ \mathcal{C}^{A,(12)(34),-}_{\bar{\mathbf{M}}})$ with the latter normalized as $c_{\rm NS} = -i$. Assume the pair $(\mathcal{B}^{\sigma}_{\mathbf{L},\mathbf{M}}, \mathcal{C}^{\pi}_{\bar{\mathbf{M}}+\mathbf{2\nu}})$ satisfy the condition $(3)_{\rm I} \times (1)_{\rm I}$ of PIB 2, namely

$$M_{12} = \frac{1}{2}(\bar{M}_1 + \bar{M}_2) + 2\nu \mod 8,$$
$$M_{345} = \frac{1}{2}(\bar{M}_3 + \bar{M}_4 + \bar{M}_5) + 3\nu \mod 4.$$

The gauge group on branes with generic $\mathbf{L}$ is either Sp or O depending on whether ν is even or odd. For special $\mathbf{L}$, namely ($L_{12} = 3,\ L_{345} = 1$) they break into short-orbit branes supporting a unitary gauge group.

We next consider some type IIB examples:

- Consider a non-permuted brane $\mathcal{B}^{B,\mathrm{id}}_{\mathbf{L},\mathbf{M}}$ invariant under the orientifold $\mathcal{C}^{B,\mathrm{id},\epsilon_1\cdots\epsilon_5}_{\bar{\mathbf{M}}}$. We normalize the orientifold by setting $c_{\mathrm{NS}} \;=\; -i^{\alpha}$, where α is the number of ϵ_a's taking minus sign. The $\mathbf{L}$-label of branes is called generic if $L_a = k_a/2$ for at most one a. If a brane $\mathcal{B}^{B,\mathrm{id}}_{\mathbf{L},\mathbf{M}}$ with generic $\mathbf{L}$ is invariant under the orientifold $\mathcal{C}^{B,\mathrm{id},\epsilon_1\cdots\epsilon_5}_{\bar{\mathbf{M}}}$, then there is a set of integer $\{\nu_a\}$ such that $\mathcal{B}^{\mathrm{id}}_{\mathbf{L},\mathbf{M}}$ and $\mathcal{C}^{\mathrm{id}}_{\bar{\mathbf{M}}+\mathbf{2}\vec{\nu}}$ satisfy the condition PIB 2. The NS parity eigenvalue is then given by
$$(-)^{F_L}P \;=\; -\,i^{\alpha+\sharp(1)_{\mathrm{II}}}\prod_a \epsilon_a^{\nu_a} \;=\; -\,i^{\alpha+\sharp(1)_{\mathrm{II}}}\prod_a \epsilon_a^{L_a}\cdot\prod_a \epsilon_a^{\bar{M}_a/2}. \tag{6.24}$$
Here we used that $L_a + M_a$ and $\frac{k_a+2}{2}$ are even for all a. Note also that $\alpha + \sharp(1)_{\mathrm{II}}$ is always even if the brane and orientifold preserve the same supersymmetry. The branes with $p(\geq 2)$ of L_a's coinciding with $\frac{k_a}{2}$ are special. The NS parity eigenvalue for such branes is determined by applying the general formula (6.23),
$$(-)^{F_L}P \;=\; -\,\mathrm{sgn}\left[\mathrm{Re}\left(i^{\alpha}(1+i)^p\right)\right]\prod_a \epsilon_a^{L_a}\cdot\prod_a \epsilon_a^{\bar{M}_a/2}. \tag{6.25}$$
We thus recover the result of tables 9,10 of [12]. The gauge group is unitary when p is even and $\alpha + \frac{p}{2}$ is an odd integer.

6.3 Tadpole cancellation

Here we discuss the RR tadpole cancellation condition and its solutions. The formula relating the charges of crosscaps and boundary states in minimal models allows us to find a set of D-branes cancelling the RR-charge of any given orientifold. It is more difficult to find the set of D-branes preserving a spacetime supersymmetry. In principle we have to deal with a system of coupled linear equations with integer coefficients, and the complexity of the problem depends on the number of linear equations which equals the dimension of the RR-charge lattice.

6.3.1 Type IIA on (55555)

There are three physically inequivalent orientifolds, $\mathcal{C}^{\mathrm{id}}_{\mathbf{0}}$, $\mathcal{C}^{(12)}_{\mathbf{0}}$ and $\mathcal{C}^{(12)(34)}_{\mathbf{0}}$. We only consider those with negative tension (O^--planes). These three orientifolds have supersymmetry phase $\varphi = 0, 1/2, 1$ respectively. The simplest tadpole-free configurations for these orientifolds are obtained by wrapping four D-branes of the like charge, same supersymmetry phase on top of the orientifolds. Such configurations are described by the tadpole states,
$$|\mathcal{C}^{\mathrm{id}}_{\mathbf{0}}\rangle + 4|\mathcal{B}^{\mathrm{id}}_{\substack{\mathbf{L}=(22222)\\ \mathbf{M}=(22222)}}\rangle,\quad |\mathcal{C}^{(12)}_{\mathbf{0}}\rangle + 4|\mathcal{B}^{(12)}_{\substack{\mathbf{L}=(0222)\\ \mathbf{M}=(9222)}}\rangle,\quad |\mathcal{C}^{(12)(34)}_{\mathbf{0}}\rangle + 4|\mathcal{B}^{(12)(34)}_{\substack{\mathbf{L}=(002)\\ \mathbf{M}=(992)}}\rangle. \tag{6.26}$$
These will be all interpreted as four D6-branes on top of orientifold plane wrapping an $\mathbb{RP}^3$ [7], and supporting $O(4)$ gauge theory with various matters.

6.3.2 Type IIA on (88444)

We have found 30 physically inequivalent orientifolds labelled by different choices of $(\pi, \mathbf{M})$ (5.40) and a sign ϵ. The choice

$$c_{\rm RR} = -1, \qquad c_{\rm NSNS} = -1 \ (\epsilon = 1) \ \ \text{or} \ \ -i \ (\epsilon = -1)$$

ensures the negative semi-definiteness of the tension for all choices of $(\pi, \mathbf{M})$ in the list. For 12 of them labelled by $\pi = \text{id}$, one finds the expressions the RR-charges in terms of those of D-branes [12],

$$\begin{aligned}
[\mathcal{C}^{\rm id,\pm}_{(00000)}] + 2[\mathcal{B}^{\rm id}_{\substack{\mathbf{L}=(33111)\\ \mathbf{M}=(33111)}}] \mp 2[\mathcal{B}^{\rm id}_{\substack{\mathbf{L}=(33111)\\ \mathbf{M}=(33111)}}] &= 0,\\
[\mathcal{C}^{\rm id,\pm}_{(00002)}] + 2[\mathcal{B}^{\rm id}_{\substack{\mathbf{L}=(33111)\\ \mathbf{M}=(33113)}}] \mp 2[\mathcal{B}^{\rm id}_{\substack{\mathbf{L}=(33111)\\ \mathbf{M}=(33111)}}] &= 0,\\
[\mathcal{C}^{\rm id,\pm}_{(02000)}] + 2[\mathcal{B}^{\rm id}_{\substack{\mathbf{L}=(33111)\\ \mathbf{M}=(35111)}}] \mp 2[\mathcal{B}^{\rm id}_{\substack{\mathbf{L}=(33111)\\ \mathbf{M}=(33111)}}] &= 0,\\
[\mathcal{C}^{\rm id,\pm}_{(02002)}] + 2[\mathcal{B}^{\rm id}_{\substack{\mathbf{L}=(33111)\\ \mathbf{M}=(35113)}}] \mp 2[\mathcal{B}^{\rm id}_{\substack{\mathbf{L}=(33111)\\ \mathbf{M}=(33111)}}] &= 0,\\
[\mathcal{C}^{\rm id,\pm}_{(22000)}] + 2[\mathcal{B}^{\rm id}_{\substack{\mathbf{L}=(33111)\\ \mathbf{M}=(55111)}}] \mp 2[\mathcal{B}^{\rm id}_{\substack{\mathbf{L}=(33111)\\ \mathbf{M}=(33111)}}] &= 0,\\
[\mathcal{C}^{\rm id,\pm}_{(22002)}] + 2[\mathcal{B}^{\rm id}_{\substack{\mathbf{L}=(33111)\\ \mathbf{M}=(55113)}}] \mp 2[\mathcal{B}^{\rm id}_{\substack{\mathbf{L}=(33111)\\ \mathbf{M}=(33111)}}] &= 0.
\end{aligned} \tag{6.27}$$

Note that each of the D-brane charges appearing above equalities expresses the sum of the charges of two short-orbit branes labelled by $\mathbf{L}, \mathbf{M}$ (recall that the non-permuted branes with $L_1 = L_2 = 3$ are fixed under γ^4). These relations immediately give RR tadpole free configurations, which are however not supersymmetric except for those in the first line. In [12], some supersymmetric tadpole-free configurations were found by rewriting these equations using the relations between D-brane charges in minimal models,

$$[\mathcal{B}_{L,M}] \;=\; [\mathcal{B}_{0,M-L}] + [\mathcal{B}_{0,M-L+2}] + \cdots + [\mathcal{B}_{0,M+L}]. \tag{6.28}$$

For some of the other 18 orientifolds, we found the following equalities for the RR charges,

$$\begin{aligned}
[\mathcal{C}^{(12),\pm}_{(00000)}] + 2[\mathcal{B}^{(12)}_{\substack{\mathbf{L}=(0111)\\ \mathbf{M}=(3333)}}] \mp 2[\mathcal{B}^{(12)}_{\substack{\mathbf{L}=(0111)\\ \mathbf{M}=(1333)}}] &= 0,\\
[\mathcal{C}^{(12),\pm}_{(00002)}] + 2[\mathcal{B}^{(12)}_{\substack{\mathbf{L}=(0111)\\ \mathbf{M}=(3335)}}] \mp 2[\mathcal{B}^{(12)}_{\substack{\mathbf{L}=(0111)\\ \mathbf{M}=(1333)}}] &= 0,\\
[\mathcal{C}^{(34),\pm}_{(00000)}] + 2[\mathcal{B}^{(34)}_{\substack{\mathbf{L}=(3301)\\ \mathbf{M}=(5533)}}] \mp 2[\mathcal{B}^{(34)}_{\substack{\mathbf{L}=(3301)\\ \mathbf{M}=(5513)}}] &= 0,\\
[\mathcal{C}^{(34),\pm}_{(02000)}] + 2[\mathcal{B}^{(34)}_{\substack{\mathbf{L}=(3301)\\ \mathbf{M}=(5733)}}] \mp 2[\mathcal{B}^{(34)}_{\substack{\mathbf{L}=(3301)\\ \mathbf{M}=(5513)}}] &= 0,\\
[\mathcal{C}^{(34),\pm}_{(22000)}] + 2[\mathcal{B}^{(34)}_{\substack{\mathbf{L}=(3301)\\ \mathbf{M}=(7733)}}] \mp 2[\mathcal{B}^{(34)}_{\substack{\mathbf{L}=(3301)\\ \mathbf{M}=(5513)}}] &= 0,\\
[\mathcal{C}^{(12)(34),\pm}_{(00000)}] + 2[\mathcal{B}^{(12)(34)}_{\substack{\mathbf{L}=(001)\\ \mathbf{M}=(333)}}] \mp 2[\mathcal{B}^{(12)(34)}_{\substack{\mathbf{L}=(001)\\ \mathbf{M}=(113)}}] &= 0.
\end{aligned} \tag{6.29}$$

Applying recombination to some of them, we found the following supersymmetric tadpole-free configurations,

$$
\begin{aligned}
&|\mathcal{C}^{(12),-}_{(00000)}\rangle + 2\sum_{\epsilon} |\mathcal{B}^{(12),\epsilon}_{\substack{\mathbf{L}\,=(1111)\\ \mathbf{M}=(2111)}}\rangle, \\
&|\mathcal{C}^{(34),-}_{(00000)}\rangle + 2\sum_{\epsilon} |\mathcal{B}^{(34),\epsilon}_{\substack{\mathbf{L}\,=(3311)\\ \mathbf{M}=(5523)}}\rangle, \\
&|\mathcal{C}^{(34),+}_{(22000)}\rangle + 2\sum_{\epsilon} |\mathcal{B}^{(34),\epsilon}_{\substack{\mathbf{L}\,=(3301)\\ \mathbf{M}=(7733)}}\rangle + 2\sum_{\epsilon} |\mathcal{B}^{(34),\epsilon}_{\substack{\mathbf{L}\,=(3321)\\ \mathbf{M}=(5553)}}\rangle.
\end{aligned}
\tag{6.30}
$$

Here ϵ specifies the characters of the stabilizer group $\mathbb{Z}_2$ of short-orbit D-branes.

The remaining 6 orientifolds all involve the permutation orientifold $|\mathcal{C}^{(12)}_{M,M+8}\rangle$ of the first two minimal models. The crosscap states are made of closed string states sitting in $\gamma^4_{(A)}$-twisted sector, and are in particular tensionless.

6.3.3 Type IIB

In type IIB Gepner models, the tadpole-free condition can be solved more easily because the charge of D-branes span a lattice of relatively low dimension.

Let us first focus on the charges arising from the untwisted sector (in the mirror description). In mirror Gepner model labelled by $(k_1 \cdots k_r)$ and $H \equiv \text{l.c.m.}(k_a+2)$, the relevant RR ground states are labelled by a mod-H integer ν which is not multiple of any of (k_a+2). They take the form

$$
|\nu\rangle_{\rm RR} \;=\; i^{-r}\bigotimes_{a=1}^{r} |(l_a, l_a+1, 1)\otimes(l_a, -l_a-1, -1)\rangle\cdot(-)^{d_a}, \tag{6.31}
$$

where (l_a, d_a) is a unique pair of integers satisfying $\nu = d_a(k_a+2)+l_a+1$. Counting the allowed ν's one finds the dimension of RR charge lattice spanned by the ground states in the untwisted sector, which is 4 for $(k_a+2)=(5,5,5,5,5)$ and 6 for $(k_a+2)=(8,8,4,4,4)$. Since the dimension agrees with the known value of $2h_{1,1}+2$ for both cases, there are no RR-charges from twisted sectors for these two theories.

The boundary states $|\mathcal{B}^{\sigma,\rho}_{\mathbf{L},\mathbf{M}}\rangle_{\rm RR+}$ are shown to have the following overlaps,

$$
{}_{\rm RR}\langle\nu|\mathcal{B}^{\sigma,\rho}_{\mathbf{L},\mathbf{M}}\rangle_{\rm RR+} \;=\; \frac{1}{2^{[p/2]}\sqrt{H}}\frac{\prod_{c=1}^{[\sigma]} F_{L_c,M_c}(\omega^{\nu w_c})(k_c+2)^{\delta_c}}{\prod_{a=1}^{r}|1-\omega^{\nu w_a}|^{1/2}}. \tag{6.32}
$$

Here we denoted $\omega \equiv e^{\frac{2\pi i}{H}}$, $w_a \equiv \frac{H}{k_a+2}$ and

$$
\begin{aligned}
F_{L,M}(x) &\equiv x^{\frac12(M+L+1)} - x^{\frac12(M-L-1)}, \\
\delta_c &\equiv \max([\tfrac{|\sigma_c|-1}{2}], 0), \\
p &\equiv (\text{number of odd-length cycles labelled by } L = k/2).
\end{aligned}
\tag{6.33}
$$

The powers of (k_c+2) and the factor $2^{[p/2]}$ arise from the order of the stabilizer group and its untwisted subgroup. The RR charge of B-branes are thus expressed conveniently by the polynomial,

$$
[\mathcal{B}^{\sigma}_{\mathbf{L},\mathbf{M}}](x) \;\equiv\; 2^{-[p/2]}\prod_{c=1}^{[\sigma]} F_{L_c,M_c}(x^{w_c})(k_c+2)^{\delta_c}. \tag{6.34}
$$

In particular, if the argument x of the polynomials is assumed to satisfy

$$1-x^H \;=\; 1+x^{w_a}+x^{2w_a}+\cdots x^{w_a(k_a+1)} \;=\; 0, \tag{6.35}$$

one can rewrite every polynomial in terms of a finite number of monomials. The number of monomials required is the same as the dimension of the (untwisted) RR-charge lattice. So $[\mathcal{B}^{\sigma}_{\mathbf{L},\mathbf{M}}](x)$ are naturally identified with vectors on the RR-charge lattice [4]. As an application of this formula, the intersection number of D-branes is computed by the index,

$$\begin{aligned} I(\mathcal{B}^{\sigma',\rho'}_{\mathbf{L}',\mathbf{M}'},\mathcal{B}^{\sigma,\rho}_{\mathbf{L},\mathbf{M}}) &\equiv {}_{\mathrm{RR}+}\langle\mathcal{B}^{\sigma',\rho'}_{\mathbf{L}',\mathbf{M}'}|e^{-i\pi J_0}q^H|\mathcal{B}^{\sigma,\rho}_{\mathbf{L},\mathbf{M}}\rangle_{\mathrm{RR}+} \\ &= \sum_{\nu} {}_{\mathrm{RR}+}\langle\mathcal{B}^{\sigma',\rho'}_{\mathbf{L}',\mathbf{M}'}|e^{-i\pi J_0}|\nu\rangle_{\mathrm{RR}+}\cdot {}_{\mathrm{RR}+}\langle\nu|\mathcal{B}^{\sigma,\rho}_{\mathbf{L},\mathbf{M}}\rangle_{\mathrm{RR}+} \\ &= \frac{1}{H}\sum_{\nu}\frac{[\mathcal{B}^{\sigma'}_{\mathbf{L}',\mathbf{M}'}](\omega^{-\nu})[\mathcal{B}^{\sigma}_{\mathbf{L},\mathbf{M}}](\omega^{\nu})}{\prod_{a=1}^{r}(1-\omega^{\nu w_a})}. \end{aligned} \tag{6.36}$$

The polynomials $[\mathcal{B}^{\sigma}_{\mathbf{L},\mathbf{M}}](x)$ satisfy various relations under the assumption (6.35). For example, for the model (55555) one finds relations among RR-charges of various permutation branes by a repeated use of the formula $(x^{\frac{1}{2}}-x^{-\frac{1}{2}})^{-1}=\frac{1}{5}(x^{-\frac{3}{2}}+2x^{-\frac{1}{2}}-2x^{\frac{1}{2}}-x^{\frac{3}{2}})$.

$$\begin{aligned} [\mathcal{B}^{(12)}_{\mathbf{0},M}] &= \frac{1}{5}\left([\mathcal{B}^{\mathrm{id}}_{\mathbf{0},M-3}]+2[\mathcal{B}^{\mathrm{id}}_{\mathbf{0},M-1}]-2[\mathcal{B}^{\mathrm{id}}_{\mathbf{0},M+1}]-[\mathcal{B}^{\mathrm{id}}_{\mathbf{0},M+3}]\right), \\ [\mathcal{B}^{(12)(34)}_{\mathbf{0},M}] \;=\; \frac{1}{5}[\mathcal{B}^{(123)}_{\mathbf{0},M}] &= \frac{1}{5}\left([\mathcal{B}^{\mathrm{id}}_{\mathbf{0},M-4}]-2[\mathcal{B}^{\mathrm{id}}_{\mathbf{0},M}]+[\mathcal{B}^{\mathrm{id}}_{\mathbf{0},M+4}]\right), \end{aligned} \tag{6.37}$$

where we used the label $M\equiv\sum_c M_c$ (mod 10) instead of $\mathbf{M}$.

It is straightforward to express the RR charge of orientifolds in terms of similar polynomials, using the relations (4.20) and (4.34). For the model (55555) one has simple relations

$$\begin{aligned} [\mathcal{C}^{\mathrm{id}}_{(00000)}] &= -4[\mathcal{B}^{\mathrm{id}}_{(22222),0}], \\ [\mathcal{C}^{(12)}_{(00000)}] &= -4[\mathcal{B}^{(12)}_{(0222),5}], \\ [\mathcal{C}^{(12)(34)}_{(00000)}] &= -4[\mathcal{B}^{(12)(34)}_{(002),0}]. \end{aligned} \tag{6.38}$$

This agrees with the result of [43] using the (twisted) Landau-Ginzburg description [44]. For the model (88444), there are orientifolds labelled by $(\pi,\mathbf{M})$ as well as ϵ's and r's as explained in Example 2 of section 5.2.2. Restricting to those with $r=r'=0$, the RR-charges are given by the following polynomials:

$$\begin{aligned} [\mathcal{C}^{\mathrm{id},\epsilon_1\epsilon_2\epsilon_3\epsilon_4\epsilon_5}_{(00000)}](x) &= -[\mathcal{B}^{\mathrm{id}}_{(33111),-4}](x)\cdot(1+\epsilon_1\epsilon_2x)(1+\epsilon_3x)(1+\epsilon_4x)(1+\epsilon_5x), \\ [\mathcal{C}^{\mathrm{id},\epsilon_1\epsilon_2\epsilon_3\epsilon_4\epsilon_5}_{(20000)}](x) &= -[\mathcal{B}^{\mathrm{id}}_{(33111),-2}](x)\cdot(1+\epsilon_1\epsilon_2)(1+\epsilon_3x)(1+\epsilon_4x)(1+\epsilon_5x), \\ [\mathcal{C}^{(12),\epsilon_1\epsilon_3\epsilon_4\epsilon_5}_{(00000)}](x) &= -[\mathcal{B}^{(12)}_{(0111),5}](x)\cdot(1+\epsilon_1\epsilon_3x)(1+\epsilon_1\epsilon_4x)(1+\epsilon_1\epsilon_5x), \\ [\mathcal{C}^{(12),\epsilon_1\epsilon_3\epsilon_4\epsilon_5}_{(20000)}](x) &= 0, \\ [\mathcal{C}^{(34),\epsilon_1\epsilon_2\epsilon_5}_{(00000)}](x) &= -2[\mathcal{B}^{(34)}_{(3301),6}](x)\cdot(1+\epsilon_1\epsilon_2x)(1+\epsilon_5x), \\ [\mathcal{C}^{(34),\epsilon_1\epsilon_2\epsilon_5}_{(20000)}](x) &= -2[\mathcal{B}^{(34)}_{(3301),8}](x)\cdot(1+\epsilon_1\epsilon_2)(1+\epsilon_5x), \\ [\mathcal{C}^{(12)(34),\epsilon_1\epsilon_5}_{(00000)}](x) &= -2[\mathcal{B}^{(12)(34)}_{(001),-1}](x)\cdot(1+\epsilon_1\epsilon_5x) \\ [\mathcal{C}^{(12)(34),\epsilon_1\epsilon_5}_{(20000)}](x) &= 0. \end{aligned} \tag{6.39}$$

RR charges from twisted sectors. Finally we briefly discuss the case where the RR charge lattice is not entirely spanned by the states in the untwisted sector. We take as an example the model (44666), $H = 12$. The RR charge lattice is known to be 14 dimensional, of which 8 arise from the states $|\nu\rangle_{\rm RR}$ in the untwisted sector defined at (6.31). The values $\nu = 0, 4, 6, 8$ (mod 12) are excluded, but for $\nu = 4, 8$ there are RR vacua of the form

$$\begin{aligned} |\mu, \tilde{\nu}\rangle_{\rm RR} &= |(\mu-1, \mu, 1) \otimes (\mu-1, \mu, 1)\rangle \otimes |(\mu-1, -\mu, -1) \otimes (\mu-1, -\mu, -1)\rangle \\ &\quad \otimes \prod_{a=3}^{5} |(\tilde{\nu}-1, \tilde{\nu}, 1) \otimes (\tilde{\nu}-1, -\tilde{\nu}, -1)\rangle, \\ \tilde{\nu} &\equiv \nu \bmod 6 \;=\; 4 \text{ or } 2, \quad \mu \;=\; 1, 2, 3. \end{aligned} \tag{6.40}$$

These 6 RR states from twisted sectors complete the full set of RR charges. They are sitting in the $(\gamma_1^{\mu}\gamma_2^{-\mu})$-twisted sector of the mirror Gepner model.

The B-type permutation orientifolds of the model (44666) have twisted RR-charges if π permutes 1 and 2. The permutation B-branes have twisted RR-charges if their untwisted stabilizer group contains elements $\gamma_1^{\mu}\gamma_2^{-\mu}$. The RR-charges of these branes and orientifolds are again conveniently expressed by polynomials of ($y \equiv e^{\frac{2\pi i \mu}{4}}$, $z \equiv e^{\frac{2\pi i \tilde{\nu}}{6}}$) which therefore satisfy

$$1 + y + y^2 + y^3 \;=\; 1 + z + z^2 \;=\; 0.$$

The branes carrying the twisted RR-charges are

$$\begin{aligned} [\mathcal{B}^{(12),\rho}_{L,M} \otimes \mathcal{B}^{\sigma,\rho'}_{\mathbf{L}',\mathbf{M}'}] &= [\mathcal{B}^{(12)}_{L,2\rho}](y)[\mathcal{B}^{\sigma}_{\mathbf{L}',\mathbf{M}'}](z), \\ [\mathcal{B}^{(1)(2),\pm}_{\mathbf{L}=(11),\mathbf{M}} \otimes \mathcal{B}^{\sigma,\rho'}_{\mathbf{L}',\mathbf{M}'}] &= (1 - y + y^2 - y^3)[\mathcal{B}^{\sigma}_{\mathbf{L}',\mathbf{M}'}](z). \end{aligned} \tag{6.41}$$

In the second line, none of L'_c equals 2 because otherwise the untwisted stabilizer of the brane would not contain $\eta_1\eta_2$. The orientifolds carrying the twisted RR-charges are

$$\begin{aligned} &\mathcal{C}^{B,(12),\rho}_{\mathbf{M}} &&: \rho_{r,\epsilon_1,\epsilon_3,\epsilon_4,\epsilon_5}(\vec{\nu}) = \omega_4^{-r(\nu_1-\nu_2)}\epsilon_1^{\nu_1}\epsilon_3^{\nu_3}\epsilon_4^{\nu_4}\epsilon_5^{\nu_5} \\ &\mathcal{C}^{B,(12)(34),\rho}_{\mathbf{M}} &&: \rho_{r,r',\epsilon_1,\epsilon_5}(\vec{\nu}) = \omega_4^{-r(\nu_1-\nu_2)}\omega_6^{-r'(\nu_3-\nu_4)}\epsilon_1^{\nu_1}\epsilon_5^{\nu_5}. \end{aligned}$$

We restrict to those with $\mathbf{M} = (00000)$ or (20000) and $\epsilon_1 = +1$ since all the others are related to them by symmetries. Their twisted RR-charges are expressed by the polynomials

$$\begin{aligned} [\mathcal{C}^{B,(12),\rho}_{(M0000)}] &= -2\left([\mathcal{B}^{(12)}_{0,-2r}](y) + (-)^{M/2}[\mathcal{B}^{(12)}_{0,-2r+4}](y)\right) \\ &\quad \times \frac{1}{4}[\mathcal{B}^{\rm id}_{(222),6}](z)(1+\epsilon_3 z^2)(1+\epsilon_4 z^2)(1+\epsilon_5 z^2), \\ [\mathcal{C}^{B,(12)(34),\rho}_{(M0000)}] &= -2\left([\mathcal{B}^{(12)}_{0,-2r}](y) + (-)^{M/2}[\mathcal{B}^{(12)}_{0,-2r+4}](y)\right) \\ &\quad \times \frac{1}{2}[\mathcal{B}^{(34)}_{(02),1}](z)(1+\epsilon_5 z^2). \end{aligned} \tag{6.42}$$

7. Concluding remarks

In this paper we discussed the construction of permutation orientifolds in general RCFTs and then studied those in Gepner models. Although our analysis was limited to the Gepner point, it will serve as a starting point to explore a new class of four-dimensional string vacua. It will be interesting to see how various properties of permutation orientifolds continue in moduli space to large volume. In doing this, it will be useful to switch from the description in terms of coset CFTs to those in terms of Landau-Ginzburg orbifolds or linear sigma models. A number of works along this path have appeared recently [44, 43, 37].

Acknowledgments

This work is a continuation of the author's previous work in collaboration with I. Brunner, K. Hori and J. Walcher; it is a preasure to thank them for discussions and correspondences. The author especially thanks K. Hori for collaboration at an early stage, and I. Brunner for informing about the work on closely related problems.

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